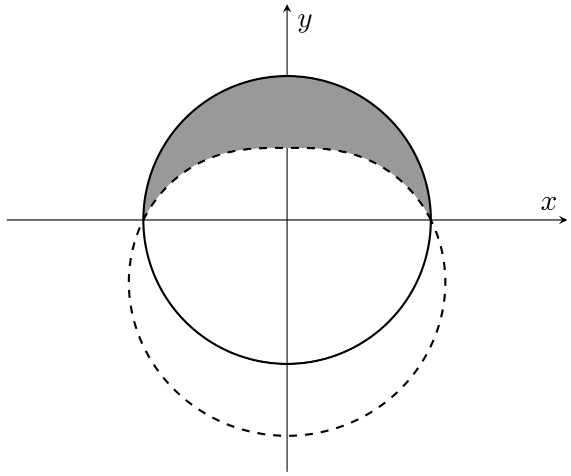
7. [10 points] Nowadays, David's haircut is more simplistic. In a poster for a more recent album, David's haircut is given by the shaded region below. The solid curve is a circle centered at the origin with radius 2, while the dashed curve is given by the equation $r(\theta) = 2 - \sin(\theta)$, where $0 \leq \theta \leq 2\pi$.



a. [2 points] Find the two intersection points of the two curves in the diagram. Express the intersection points in polar coordinate form (r, θ), where $r > 0$ and $0 \leq \theta < 2\pi$.

Solution: The intersection points correspond to $2 - \sin(\theta) = 2$. Solving, we find $\theta = 0$, and $\theta = \pi$ since we require $0 \leq \theta < 2\pi$.

Answer: $(2, 0)$ and $(2, \pi)$

b. [4 points] Write an expression involving one or more integrals that gives the total perimeter of the shaded region. Your final answer should not involve the letter r. Do not evaluate your integral(s).

Solution: The solid line is the arc of a semicircle with radius 2, so has length 2π.
For the dotted curve, we note that $r'(\theta) = -\cos(\theta)$, so the arclength is

$$\int_0^\pi \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta = \int_0^\pi \sqrt{(2 - \sin\theta)^2 + \cos^2\theta}\, d\theta.$$

Therefore the total perimeter of the shaded area is

$$2\pi + \int_0^\pi \sqrt{(2 - \sin\theta)^2 + \cos^2\theta}\, d\theta.$$

Answer: $2\pi + \int_0^\pi \sqrt{(2 - \sin\theta)^2 + \cos^2\theta}\, d\theta$

c. [4 points] Write an expression involving one or more integrals that gives the area of the shaded region. Your final answer should not involve the letter r. Do not evaluate your integral(s).

Solution:
The area inside the dotted curve but above the x-axis is given by $\frac{1}{2}\int_0^\pi (2 - \sin\theta)^2\, d\theta$. We subtract this from the area of the semi-circle above the x-axis to obtain

$$2\pi - \frac{1}{2}\int_0^\pi (2 - \sin\theta)^2\, d\theta.$$

Answer: $2\pi - \frac{1}{2}\int_0^\pi (2 - \sin\theta)^2\, d\theta$