9. [11 points] Consider the function

$$g(x) = \frac{4}{\sqrt{1+5x^3}}.$$

a. [5 points] Give the first three nonzero terms of the Taylor series of $g(x)$ centered about $x = 0$. Show all your work.

Solution: We use the binomial series

$$(1+y)^{-1/2} = 1 - \frac{1}{2}y + \frac{3}{8}y^2 + \cdots$$

where we replace y with $5x^3$, and multiply by 4. We find that the first three terms of the Taylor series of $g(x)$ centered at 0 are:

$$4 - \frac{4}{2}(5x^3) + \frac{12}{8}(5x^3)^2 = 4 - 10x + \frac{75}{2}x^6.$$

Answer: $4 - 10x^3 + \frac{75}{2}x^6$

b. [2 points] What is the radius of convergence of the Taylor series for $g(x)$ around $x = 0$?

Solution: The binomial series converges for $|y| < 1$, so to figure out the radius of convergence of the Taylor series for $g(x)$, we look for where

$$|5x^3| < 1$$

which gives us

$$|x| < \left(\frac{1}{5}\right)^{1/3}.$$

Answer: $\left(\frac{1}{5}\right)^{1/3}$

c. [4 points] Suppose that $G(x)$ is an antiderivative for $g(x)$ which satisfies $G(0) = 5$. Give the first four non-zero terms of the Taylor series for $G(x)$ centered about $x = 0$.

Solution: We use the first three non-zero terms of the Taylor series of $g(x)$ we got in **a.** Taking the indefinite integral, we get

$$C + 4x - \frac{10}{4}x^4 + \frac{75}{14}x^7.$$

Since $G(0) = 5$, we get that the first four non-zero terms of the Taylor series of $G(x)$ are

$$5 + 4x - \frac{5}{2}x^4 + \frac{75}{14}x^7.$$

Answer: $5 + 4x - \frac{5}{2}x^4 + \frac{75}{14}x^7$