1. [9 points] When a rocket leaves the gravitational influence of the Earth, it could travel infinitely far away (if we ignore the effects of other celestial bodies). When a rocket of mass m kilograms is h meters above the surface of the Earth, it has a weight of $w = 9.8m\left(\frac{6{,}400{,}000}{6{,}400{,}000+h}\right)^2$ Newtons. Here, $6,400,000$ is the radius of the Earth in meters, and 9.8 is the gravitational constant in $\mathrm{m/s^2}$.

a. [3 points] Approximately how much work is required to lift the rocket Δh additional meters when it is already h meters above the surface of the Earth? Your answer may include m, h, and Δh.

Solution: The weight of the rocket is $9.8m\left(\frac{6{,}400{,}000}{6{,}400{,}000+h}\right)^2$, and so the work needed is

$$9.8m\left(\frac{6,400,000}{6,400,000+h}\right)^2 \Delta h \text{ Joules.}$$

b. [6 points] Figure out the work required to lift the rocket from the surface of the Earth to a height of infinity. Your answer may include m.

Solution: We integrate as h goes between 0 and ∞ :

$$\begin{aligned}\int_0^\infty 9.8m\left(\frac{6,400,000}{6,400,000+h}\right)^2 dh &= 9.8m \lim_{b\to\infty}\int_0^b \left(\frac{6,400,000}{6,400,000+h}\right)^2 dh\\ &= 9.8m(6,400,000)^2 \lim_{b\to\infty}\int_{6{,}400{,}000}^{b+6{,}400{,}000} \frac{du}{u^2}\\ &= 9.8(6,400,000)^2 m \lim_{b\to\infty}\left[\frac{1}{b+6,400,000} - \frac{1}{6,400,000}\right]\\ &= 9.8(6,400,000)m \text{ Joules.}\end{aligned}$$