3. [8 points] Let

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^3}(x-2)^n$$

Find the interval of convergence of the power series. Justify your answer.

Solution: By the ratio test, the interval of convergence (except for the endpoints) consists of those x-values for which

$$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{4^{n+1}(n+1)^3}}{\frac{(x-2)^n}{4^n n^3}} \right| < 1$$

Simplifying, we get

$$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{4^{n+1}(n+1)^3}}{\frac{(x-2)^n}{4^n n^3}} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{(x-2)n^3}{4(n+1)^3} \right| < 1$$

$$|x-2| \lim_{n\to\infty} \left| \frac{n^3}{4(n^3+3n^2+3n+1)} \right| < 1$$

$$|x-2|\frac{1}{4} < 1$$

$$|x-2| < 4$$

so the tentative interval of convergence is $-2 < x < 6$. We must check the endpoints

- **Endpoint** $x = -2$**:**

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^3}(-2-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

converges by the alternating series test.

- **Endpoint** $x = 6$**:**

$$\sum_{n=1}^{\infty} \frac{1}{4^n n^3}(6-2)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges by the p-test.

Therefore, the interval of convergence is $-2 \le x \le 6$.