9. [9 points]

a. [2 points] Find the Taylor series about $x = 0$ of $\sin(x^2)$. Your answer should include a formula for the general term in the series.

Solution:

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \cdots + \frac{(-1)^n\, x^{4n+2}}{(2n+1)!} + \cdots$$

b. [2 points] Let m be a positive integer, find the Taylor series about $x = 0$ of $\cos(m\pi x)$. Your answer should include a formula for the general term in the series.

Solution:

$$\cos(m\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (m\pi x)^{2n}}{(2n)!} = 1 - \frac{m^2\pi^2 x^2}{2!} + \cdots + \frac{(-1)^n (m\pi x)^{2n}}{(2n)!} + \cdots$$

c. [5 points] Use the second degree Taylor polynomials of $\sin(x^2)$ and $\cos(m\pi x)$ to approximate the value of b_m, where

$$b_m = \int_{-1}^{1} \sin(x^2)\cos(m\pi x)dx.$$

(The number b_m is called a *Fourier coefficient of the function* $\sin x^2$. These numbers play a key role in *Fourier analysis*, a subject with widespread applications in engineering and the sciences.)

Solution:

$$b_m \approx \int_{-1}^{1} x^2\left(1 - \frac{m^2\pi^2x^2}{2!}\right)dx$$

$$b_m \approx \int_{-1}^{1} x^2 - \frac{m^2\pi^2}{2}x^4dx$$

$$b_m \approx \frac{x^3}{3} - \frac{m^2\pi^2}{10}x^5|_{-1}^{1}$$

$$b_m \approx \frac{2}{3} - \frac{m^2\pi^2}{5}$$