10. [9 points] A patient takes a drug in doses of 100 mg once every 24 hours. The half-life of the drug in the patient's body is 12 hours. Let D_n be the amount of the drug in the patient immediately after taking the nth dose of the drug. Be sure to include units.

a. [3 points] Find D_1, D_2 and D_3.

Solution: Since the half-life is 12 hours, after 24 hours, $\frac{1}{4}$ of the drug remains in the body.

$$D_1 = 100\text{mg}$$
$$D_2 = 100 + 100\left(\tfrac{1}{4}\right) = 100\left(1 + \tfrac{1}{4}\right) = 125\text{mg}$$
$$D_3 = 100\left(1 + \tfrac{1}{4} + \tfrac{1}{16}\right) = 100\left(1 + \tfrac{1}{4} + \tfrac{1}{16}\right) = 131.25 \text{ mg}$$

b. [4 points] Find a closed form expression (an expression that does not involve a long summation or a recursive formula) for D_n.

Solution: From part **a**

$$D_n = 100 + 100\left(\tfrac{1}{4}\right) + 100\left(\tfrac{1}{4}\right)^2 + \cdots + 100\left(\tfrac{1}{4}\right)^{n-1}.$$

This is a finite geometric series with first term 100 and the common ratio between terms is $\frac{1}{4}$. So we have

$$D_n = \sum_{k=1}^{n} 100\left(\tfrac{1}{4}\right)^{k-1} = \frac{100\left(1 - \left(\tfrac{1}{4}\right)^n\right)}{\frac{3}{4}} = \frac{400}{3}\left(1 - \left(\tfrac{1}{4}\right)^n\right) \text{ mg}$$

c. [2 points] What is $\lim_{n\to\infty} D_n$?

Solution: Taking the limit, we obtain

$$\lim_{n\to\infty} \frac{400}{3}\left(1 - \left(\tfrac{1}{4}\right)^n\right) = \frac{400}{3} \approx 133.3 \text{ mg}$$