10. [9 points] A patient takes a drug in doses of 100 mg once every 24 hours. The half-life of the drug in the patient’s body is 12 hours. Let $D_n$ be the amount of the drug in the patient immediately after taking the $n$th dose of the drug. Be sure to include units.

a. [3 points] Find $D_1$, $D_2$ and $D_3$.

**Solution:** Since the half-life is 12 hours, after 24 hours, $\frac{1}{4}$ of the drug remains in the body.

\[
D_1 = 100 \text{mg}
\]

\[
D_2 = 100 + 100 \left( \frac{1}{4} \right) = 100 \left( 1 + \frac{1}{4} \right) = 125 \text{mg}
\]

\[
D_3 = 100 \left( 1 + \frac{1}{4} + \frac{1}{16} \right) = 100 \left( 1 + \frac{1}{4} + \frac{1}{16} \right) = 131.25 \text{ mg}
\]

b. [4 points] Find a closed form expression (an expression that does not involve a long summation or a recursive formula) for $D_n$.

**Solution:** From part a

\[
D_n = 100 + 100 \left( \frac{1}{4} \right) + 100 \left( \frac{1}{4} \right)^2 + \cdots + 100 \left( \frac{1}{4} \right)^{n-1}.
\]

This is a finite geometric series with first term 100 and the common ratio between terms is $\frac{1}{4}$. So we have

\[
D_n = \sum_{k=1}^{n} 100 \left( \frac{1}{4} \right)^{k-1} = \frac{100 \left( 1 - \left( \frac{1}{4} \right)^n \right)}{\frac{3}{4}} = \frac{400}{3} \left( 1 - \left( \frac{1}{4} \right)^n \right) \text{ mg}
\]

c. [2 points] What is $\lim_{n \to \infty} D_n$?

**Solution:** Taking the limit, we obtain

\[
\lim_{n \to \infty} \frac{400}{3} \left( 1 - \left( \frac{1}{4} \right)^n \right) = \frac{400}{3} \approx 133.3 \text{ mg}
\]