3. [12 points]

a. [6 points] State whether each of the following series converges or diverges. Indicate which test you use to decide. Show all of your work to receive full credit.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution: The function $\frac{1}{n\sqrt{\ln n}}$ is decreasing and positive for $n \geq 2$, then the Integral test says that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ behaves as $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx$.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln x}}dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-\frac{1}{2}}du = \lim_{b\to\infty} 2\sqrt{u}\,\Big|_{\ln 2}^{\ln b} = \infty.$$

Hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$

Solution: Since $0 \leq \dfrac{\cos^2(n)}{\sqrt{n^3}} \leq \dfrac{1}{n^{\frac{3}{2}}}$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by p-series test ($p = \frac{3}{2} > 1$), then comparison test yields the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$.

b. [6 points] Decide whether each of the following series converges absolutely, converges conditionally or diverges. Circle your answer. No justification required.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$

Converges absolutely **Converges conditionally** Diverges

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{3n}}{5^n}$

Converges absolutely Converges conditionally **Diverges**

Solution: $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right| = \sum_{n=0}^{\infty}\frac{\sqrt{n^2+1}}{n^2+n+8}$ behaves as $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ since

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n^2+1}}{n^2+n+8}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n\sqrt{n^2+1}}{n^2+n+8} = 1 > 0.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (p-series test $p=1$), then by limit comparison test $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right|$ diverges.

The convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$ follows from alternating series test since for $a_n = \frac{\sqrt{n^2+1}}{n^2+n+8}$:

- $\displaystyle\lim_{n\to\infty} a_n = 0$.
- a_n is decreasing

$$\frac{d}{dn}\left(\frac{\sqrt{n^2+1}}{n^2+n+8}\right) = \frac{-1+6n-n^3}{\sqrt{1+n^2}(n^2+n+8)^2} < 0$$

for n large.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^{3n}}{5^n} = \sum_{n=0}^{\infty}\left(-\frac{8}{5}\right)^n$ is a geometric series with ratio $r = -\frac{8}{5} < -1$, hence it diverges.