5. [8 points] Consider

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n}.$$

a. [2 points] Does the series converge for $x = 2$? Justify your answer.

Solution: At $x = 2$

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n} = \sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, 4^n = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

The series diverge since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

b. [2 points] Based only on your answer from part **a**, what can you say about R, the radius of convergence of the series? Circle your answer.

Solution: $R = 2$ $\quad R > 2$ $\quad R < 2$ $\quad \boxed{R \leq 2}$ $\quad R \geq 2$

since it is possible for $x = 2$ to be one of the endpoints in the interval of convergence.

c. [4 points] Find the interval of convergence of the series.

Solution:

$$\lim_{n\to\infty} \frac{\left|\frac{n+1}{4^{n+1}(n+2)}\, x^{2n+2}\right|}{\left|\frac{n}{4^n(n+1)}\, x^{2n}\right|} = x^2 \lim_{n\to\infty} \frac{(n+1)^2}{4n(n+2)} = \frac{x^2}{4}.$$

If $\frac{x^2}{4} < 1$, then the series converges. Hence Ratio test states that the series converges if $-2 < x < 2$. We need to check the endpoints $x = \pm 2$. We already checked $x = 2$. For $x = -2$,

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n} = \sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, 4^n = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

diverges by part **a**. The interval of convergence of the series is $-2 < x < 2$.