8. [8 points] The function

$$F(x) = \int_0^x \sqrt{1+9t^4}dt.$$

computes the arc length of the graph of the function $y = t^3$ from $t = 0$ to $t = x$.

a. [4 points] Approximate the value of $F(\frac{1}{2})$, the arc length of the curve $y = t^3$ for $0 \le t \le \frac{1}{2}$, using $RIGHT(2)$, $LEFT(2)$, $TRAP(2)$ and $MID(2)$. Write each term of each sum to receive full credit.

Solution: We have $\Delta x = 1/4$. Using this we have

$$LEFT(2) = \tfrac{1}{4}\left(1 + \sqrt{1 + 9\left(\tfrac{1}{4}\right)^4}\right) \approx 0.504.$$

$$RIGHT(2) = \tfrac{1}{4}\left(\sqrt{1 + 9\left(\tfrac{1}{4}\right)^4} + \sqrt{1 + 9\left(\tfrac{1}{2}\right)^4}\right) \approx 0.566.$$

$$TRAP(2) = \tfrac{1}{2}(LEFT(2) + RIGHT(2)) \approx 0.535.$$

$$MID(2) = \tfrac{1}{4}\left(\sqrt{1 + 9\left(\tfrac{1}{8}\right)^4} + \sqrt{1 + 9\left(\tfrac{3}{8}\right)^4}\right) \approx 0.521.$$

b. [2 points] Which approximation $RIGHT$ or $LEFT$ is guaranteed to give an underestimate for $F(\frac{1}{2})$? Justify.

Solution: The integrand $f(x) = \sqrt{1+9x^4}$ is an increasing function, so $LEFT(2)$ is an underestimate.

c. [2 points] Find $F'(1)$.

Solution: Using the second fundamental theorem of calculus, we have

$$F'(x) = \sqrt{1+9x^4}$$

so $F'(1) = \sqrt{10}$.