9. [10 points] A second way to approximate the function

$$F(x) = \int_0^x \sqrt{1+9t^4}dt.$$

is by using its Taylor polynomials.

a. [2 points] Find the first three nonzero terms in the Taylor series for the function $\sqrt{1+u}$ about $u = 0$.

Solution:

$$\sqrt{1+u} \approx 1 + \frac{1}{2}u - \frac{1}{8}u^2$$

b. [2 points] Find the first three nonzero terms in the Taylor series for $\sqrt{1+9t^4}$ about $t = 0$.

Solution: Let $u = 9t^4$ then

$$\sqrt{1+9t^4} \approx 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8$$

c. [2 points] Find the first three nonzero terms in the Taylor series for $F(x)$ about $x = 0$.

Solution:

$$F(x) \approx \int_0^x 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8 dt = x + \frac{9}{10}x^5 - \frac{9}{8}x^9$$

d. [2 points] For which values of x do you expect the Taylor series for $F(x)$ about $x = 0$ to converge? Justify your answer.

Solution: We substituted $u = 9t^4$ into the Binomial series. The interval of convergence for the Binomial series is $-1 < u < 1$. Then we expect the series to converge for $0 \le 9x^4 < 1$. Hence the Taylor series for $F(x)$ about $x = 0$ converges if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

e. [2 points] Use the fifth degree Taylor polynomial for $F(x)$ about $x = 0$ to approximate the value of $F(\frac{1}{2})$.

Solution: $P_5(x) = x + \frac{9}{10}x^5$, then $F\left(\frac{1}{2}\right) \approx P_5\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{9}{10}\left(\frac{1}{2}\right)^5 = 0.528$