10. [9 points]

a. [3 points] Find the first three nonzero terms in the Taylor series of \( f(y) = \frac{1}{(1 + y)^{3/2}} \) about \( y = 0 \). Show all your work.

**Solution:** Using the binomial expansion, this is

\[
\frac{1}{(1 + y)^{3/2}} \approx 1 - \frac{3}{2}y + \frac{(-3/2) \cdot (-5/2)}{2} y^2 = 1 - \frac{3}{2}y + \frac{15}{8}y^2
\]

b. [2 points] Use your answer in (a) to find the first three nonzero terms in the Taylor series of \( g(x) = \frac{1}{(a^2 + x^2)^{3/2}} \) about \( x = 0 \). Show all your work.

**Solution:** Factoring, we have

\[
\frac{1}{(a^2 + x^2)^{3/2}} = \frac{1}{(a^2(1 + (\frac{x}{a})^2))^{3/2}} = \frac{1}{(a^2)^{3/2}(1 + (\frac{x}{a})^2)^{3/2}} = \frac{1}{a^3(1 + (\frac{x}{a})^2)^{3/2}}.
\]

Therefore, letting \( y = \left(\frac{x}{a}\right)^2 \), we have

\[
\frac{1}{a^3(1 + (\frac{x}{a})^2)^{3/2}} \approx \frac{1}{a^3} \left(1 - \frac{3}{2}y + \frac{15}{8}y^2\right) = \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4.
\]

\[
\frac{1}{(a^2 + x^2)^{3/2}} \approx \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4.
\]

Problem continues on the next page

c. [2 points] For which values of \( x \) is the Taylor series for \( g(x) \) about \( x = 0 \) expected to converge?

**Solution:** The binomial series in a) converges for \( |y| < 1 \). This implies that the series for \( g(x) \) converges for all values of \( x \) satisfying \( |\frac{x}{a}| < 1 \), so \(-|a| < x < |a|\).
Continuation of problem 10.

The force of gravitational attraction \( F \) between a rod of length \( 2L \) and a particle at a distance \( a \) is given by

\[
F = k \int_{0}^{L} \frac{1}{(a^2 + x^2)^{3/2}} dx,
\]

where \( k \) is a positive constant.

d. [2 points] Use your answer in (b) to obtain an approximation for the force of gravitational attraction \( F \) between the rod and the particle. Your answer should depend on the constants \( k, a \) and \( L \). Show all your work.

**Solution:** We want

\[
F = k \int_{0}^{L} \frac{1}{(a^2 + x^2)^{3/2}} dx \approx k \int_{0}^{L} \frac{1}{a^3} - \frac{3}{2a^5} x^2 + \frac{15}{8a^7} x^4 dx
\]

\[
= k \left( \frac{L}{a^3} - \frac{1}{2a^5} L^3 + \frac{3}{8a^7} L^5 \right).
\]

Hence \( F \approx \frac{kL}{a^3} - \frac{k}{2a^5} L^3 + \frac{3k}{8a^7} L^5 \).