10. [9 points]

a. [3 points] Find the first three nonzero terms in the Taylor series of $f(y) = \dfrac{1}{(1+y)^{\frac{3}{2}}}$ about $y = 0$. Show all your work.

Solution: Using the binomial expansion, this is

$$\frac{1}{(1+y)^{3/2}} \approx 1 - \frac{3}{2}y + \frac{(-3/2)\cdot(-5/2)}{2}y^2 = 1 - \frac{3}{2}y + \frac{15}{8}y^2$$

b. [2 points] Use your answer in (**a**) to find the first three nonzero terms in the Taylor series of $g(x) = \dfrac{1}{(a^2+x^2)^{\frac{3}{2}}}$ about $x = 0$. Show all your work.

Solution: Factoring, we have

$$\frac{1}{(a^2+x^2)^{\frac{3}{2}}} = \frac{1}{(a^2(1+(\frac{x}{a})^2))^{\frac{3}{2}}} = \frac{1}{(a^2)^{\frac{3}{2}}(1+(\frac{x}{a})^2)^{\frac{3}{2}}} = \frac{1}{a^3(1+(\frac{x}{a})^2)^{3/2}}.$$

Therefore, letting $y = \left(\frac{x}{a}\right)^2$, we have

$$\frac{1}{a^3(1+(\frac{x}{a})^2)^{3/2}} = \frac{1}{a^3}\left(\frac{1}{(1+y)^{3/2}}\right) \approx \frac{1}{a^3}\left(1 - \frac{3}{2}y + \frac{15}{8}y^2\right)$$

$$= \frac{1}{a^3}\left(1 - \frac{3}{2}\left(\frac{x}{a}\right)^2 + \frac{15}{8}\left(\frac{x}{a}\right)^4\right) = \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4.$$

$$\frac{1}{(a^2+x^2)^{\frac{3}{2}}} \approx \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4.$$

c. [2 points] For which values of x is the Taylor series for $g(x)$ about $x = 0$ expected to converge?

Solution: The Binomial series in a) converges for $|y| < 1$. This implies that the series for $g(x)$ converges for all values of x satisfying $|\frac{x}{a}| < 1$, so $-|a| < x < |a|$.

Problem continues on the next page

Continuation of problem 10.

The force of gravitational attraction F between a rod of length $2L$ and a particle at a distance a is given by

$$F = k\int_0^L \frac{1}{(a^2+x^2)^{\frac{3}{2}}}dx,$$

where k is a positive constant.

d. [2 points] Use your answer in (**b**) to obtain an approximation for the force of gravitational attraction F between the rod and the particle. Your answer should depend on the constants k, a and L. Show all your work.

Solution: We want

$$\begin{aligned} F &= k\int_0^L \frac{1}{(a^2+x^2)^{3/2}}dx \approx k\int_0^L \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4 dx \\ &= k\left(\frac{L}{a^3} - \frac{1}{2a^5}L^3 + \frac{3}{8a^7}L^5\right). \end{aligned}$$

Hence $F \approx \dfrac{kL}{a^3} - \dfrac{k}{2a^5}L^3 + \dfrac{3k}{8a^7}L^5$.