6. [10 points] At a hospital, a patient is given a drug intravenously at a constant rate of r mg/day as part of a new treatment. The patient's body depletes the drug at a rate proportional to the amount of drug present in his body at that time. Let $M(t)$ be the amount of drug (in mg) in the patient's body t days after the treatment started. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = r - \frac{1}{4}M \quad \text{with} \quad M(0) = 0.$$

a. [7 points] Find a formula for $M(t)$. Your answer should depend on r.

Solution: We use separation of variables

$$\frac{dM}{r - \frac{1}{4}M} = dt.$$

Using u-substition with $u = r - 1/4M, du = -1/4dM$ for the left-hand-side, we anti-differentiate:

$$-4\ln|r - \frac{1}{4}M| = t + C_1.$$

Therefore,

$$\ln|r - \frac{1}{4}M| = -t/4 + C_2$$

and

$$|r - \frac{1}{4}M| = e^{-t/4+C_2} = C_3e^{-t/4}.$$

Therefore

$$1/4M = r - C_3e^{-t/4}$$

and

$$M(t) = 4r - C_4e^{-t/4}.$$

With $M(0) = 0$, we conclude that $C_4 = 4r$, so we get $M(t) = 4r - 4re^{-t/4}$.

b. [1 point] Find all the equilibrium solutions of the differential equation.

Solution: $M = 4r$.

c. [2 points] The treatment's goal is to stabilize in the long run the amount of drug in the patient at a level of 200 mg. At what rate r should the drug be administered?

Solution: You need $4r = 200$, then $r = 50$ mg/day.