8. [8 points] Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}(x-5)^n.$$

In the following questions, you need to support your answers by stating and properly justifying the use of the test(s) or facts you used to prove the convergence or divergence of the series. Show all your work.

a. [2 points] Does the series converge or diverge at $x = 3$?

Solution: At $x = 3$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the alternating series test, since $1/\sqrt{n}$ is decreasing and converges to 0.

b. [2 points] What does your answer from part (**a**) imply about the radius of convergence of the series?

Solution: Because it converges at $x = 3$, we know that the radius of convergence $R \geq 2$.

c. [4 points] Find the interval of convergence of the power series.

Solution: Using the ratio test, we have

$$\lim_{n\to\infty} \frac{\frac{1}{2^{n+1}\sqrt{n+1}}|x-5|^{n+1}}{\frac{1}{2^n\sqrt{n}}|x-5|^n} = \frac{1}{2}|x-5| = L,$$

so the radius of convergence is 2. Now we have to check the endpoints. We know from part (a) that it converges at $x = 3$. For $x = 7$, we get $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges. Thus, the interval of convergence is $3 \leq x < 7$.