4. [9 points] We can define the Bessel function of order one by its Taylor series about $x = 0$, $J_1(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$.

a. [3 points] Compute $J_1^{(2015)}(0)$. Write your answer in exact form and do not try evaluate using a calculator.

Solution: The 2015th derivative of J_1 at $x = 0$ corresponds to 2015! times the 1007th coefficient in the Taylor series above. This gives us that $J_1^{(2015)}(0) = \dfrac{-(2015)!}{(1007)!(1008)!2^{2015}}$.

$$J_1^{(2015)}(0) = \underline{\frac{-(2015)!}{(1007)!(1008)!2^{2015}}}$$

b. [4 points] Find $P_5(x)$, the Taylor polynomial of degree 5 that approximates $J_1(x)$ near $x = 0$.

Solution: $P_5(x) = \dfrac{x}{2} - \dfrac{x^3}{16} + \dfrac{x^5}{384}$

$$P_5(x) = \underline{\frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}}$$

c. [2 points] Use the Taylor polynomial from the previous part to compute $\displaystyle\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3}$.

Solution: $\displaystyle\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \lim_{x\to 0} \frac{-\frac{x^3}{16} + \frac{x^5}{384}}{x^3} = -\frac{1}{16}$

$$\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \underline{-\frac{1}{16}}$$