4. [9 points] We can define the Bessel function of order one by its Taylor series about $x = 0$,

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}.$$

a. [3 points] Compute $J_1^{(2015)}(0)$. Write your answer in exact form and do not try evaluate using a calculator.

**Solution:** The 2015th derivative of $J_1$ at $x = 0$ corresponds to $2015!$ times the 1007th coefficient in the Taylor series above. This gives us that $J_1^{(2015)}(0) = \frac{-(2015)!}{(1007!)(1008)!2^{2015}}$.

$$J_1^{(2015)}(0) = \frac{-(2015)!}{(1007!)(1008)!2^{2015}}$$

b. [4 points] Find $P_5(x)$, the Taylor polynomial of degree 5 that approximates $J_1(x)$ near $x = 0$.

**Solution:** $P_5(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}$

$$P_5(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}$$

c. [2 points] Use the Taylor polynomial from the previous part to compute

$$\lim_{x \to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3}.$$

**Solution:**

$$\lim_{x \to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \lim_{x \to 0} \frac{-\frac{x^3}{16} + \frac{x^5}{384}}{x^3} = -\frac{1}{16}$$

$$\lim_{x \to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = -\frac{1}{16}$$