5. [10 points]

a. [5 points] Determine the **radius** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^{4n}}{n^5 (16)^n}.$$

Solution: The above series will converge for x values such that

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} < 1$$

by the ratio test. We have

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} = \frac{1}{16}|x-5|^4.$$

and so the desired inequality holds if $\frac{1}{16}|x-5|^4 < 1$. This is equivalent to $|x-5| < 2$. Thus, the radius of convergence is 2.

The radius of convergence is $\underline{2}$.

b. [5 points] The power series $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ has radius of convergence 1. Determine the **interval** of convergence for this power series.

Solution:

For $x = 1$, we get the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$. We have that $\displaystyle\lim_{n\to\infty} \frac{\frac{n+2}{n^4+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4+2n^3}{n^4+1} = 1.$

The limit comparison test then tells us that the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges if and only if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p > 1$ it converges, and so $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges.

For $x = -1$, we get the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{n^4+1}$. By the work shown above, this series converges absolutely.

The interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ is then $-1 \le x \le 1$

The interval of convergence is $\underline{[-1, 1]}$.