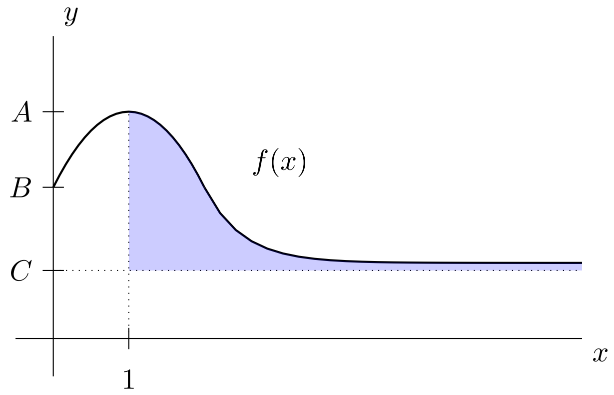
10. [14 points] A function f has domain $[0, \infty)$, and its graph is given below. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



a. [5 points] Determine the convergence of the improper integral below. **You must give full evidence supporting your answer, showing all your work and indicating any theorems about integrals you use**.

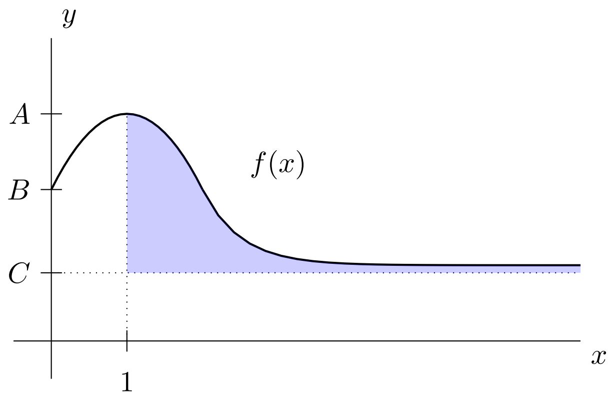
$$\int_0^1 \frac{f(x)}{x}\, dx$$

Solution: We have that for $0 < x \leq 1$

$$\frac{f(x)}{x} \geq \frac{B}{x}$$

The improper integral $\displaystyle\int_0^1 \frac{B}{x}\, dx = B\int_0^1 \frac{1}{x}\, dx$ diverges by the p-test with $p = 1$. Thus, the integral $\displaystyle\int_0^1 \frac{f(x)}{x}\, dx$ diverges by the comparison test.

10. (continued) For your convenience, the graph of f is given again. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



b. [3 points] **Circle** the correct answer. The value of the integral $\displaystyle\int_1^{\infty} f(x)f'(x)\,dx$

is $C - A$ $\quad$ $\boxed{\text{is } \frac{C^2 - A^2}{2}}$ $\quad$ is $B - A$ $\quad$ cannot be determined $\quad$ diverges

c. [3 points] **Circle** the correct answer. The value of the integral $\displaystyle\int_1^{\infty} f'(x)\,dx$

$\boxed{\text{is } C - A}$ $\quad$ is $\frac{C^2 - A^2}{2}$ $\quad$ is C $\quad$ cannot be determined $\quad$ diverges

d. [3 points] Determine, with justification, whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} (f(n) - C)$$

Solution: We notice that the function $f(x) - C$ is decreasing, positive with $\lim_{x\to\infty}(f(x) - C) = 0$. By the integral test, the series

$$\sum_{n=1}^{\infty} (f(n) - C)$$

converges if and only if the improper integral

$$\int_1^{\infty} (f(x) - C)$$

converges. But this integral gives exactly the shaded area, which we know that it is finite. So this integral converges and therefore the series converges as well.