2. [12 points] In this problem **you must give full evidence supporting your answer, showing all your work and indicating any theorems about series you use.**

a. [7 points] Show that the following series **converges**. Does it converge conditionally or absolutely? Justify.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n! + 2^n}$$

Solution: We notice that

$$\left|\frac{(-1)^n}{n! + 2^n}\right| = \frac{1}{n! + 2^n} \leq \frac{1}{2^n} \quad \text{for all } n \geq 1 \quad (*)$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges because it is a geometric series with ratio $1/2$ and $|1/2| < 1$. Thus, using the inequality $(*)$, the series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n! + 2^n}\right|$$

converges by the comparison test. Since this series converges, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n! + 2^n}$$

converges absolutely. (This also shows that it converges).

b. [5 points] Determine whether the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution: The function $f(x) = \frac{1}{x \ln x}$ is decreasing and has $\lim_{x\to\infty} f(x) = 0$. Moreover,

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\, du = \lim_{b\to\infty} (\ln(\ln b) - \ln(\ln 2)) = \infty$$

where in the second equality we used the substitution $u = \ln x$. So the integral $\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges. Therefore, the integral test applies and tells us that the series in question diverges as well.