**6**. [7 points] Consider the function $F$ defined by $F(x) = \int_{-\infty}^{x} e^{-t^2}\,dt$.

For each value of $x$, the right hand side is an improper integral that converges. The function $F(x)$ is an antiderivative of $e^{-x^2}$. (You do not need to verify this.)

**a**. [4 points] It can be shown that $F(0) = \frac{\sqrt{\pi}}{2}$. Using this fact, write the first four nonzero terms of the Taylor series for the function $F(x)$ centered at $x = 0$.

$F(x)$ $\qquad x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots$ ← this is an antideriv of $e^{-x^2}$. So shift it to make it go through $(0, \sqrt{\pi}/2)$:

$\frac{d}{dx}\downarrow \qquad \uparrow \int dx$

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$

$$\boxed{F(x) \approx \frac{\sqrt{\pi}}{2} + x - \frac{1}{3}x^3 + \frac{1}{10}x^5}$$

**b**. [3 points] Use your answer from part **a.** to approximate the integral $\int_0^1 e^{-t^2}\,dt$.

$$\int_0^1 e^{-t^2}\,dt = F(1) - F(0) \approx \left[\frac{\sqrt{\pi}}{2} + 1 - \frac{1}{3} + \frac{1}{10}\right] - \frac{\sqrt{\pi}}{2}$$

$$= 1 - \frac{1}{3} + \frac{1}{10} = \frac{30}{30} - \frac{10}{30} + \frac{3}{30} = \boxed{\frac{23}{30}}$$

**7**. [6 points]

**a**. [3 points] Find the interval of convergence of the power series $\sum_{k=3}^{\infty} \frac{x^k}{(k^2+1)3^k}$.

You do not need to show your work.

Ratio Test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{((k+1)^2+1)3^{k+1}} \cdot \frac{(k^2+1)3^k}{x^k} \right|$$

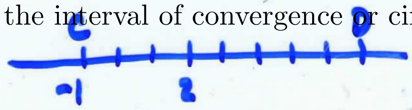
$$= \lim_{k\to\infty} \left|\frac{x^{k+1}}{x^k}\right| \cdot \frac{3^k}{3^{k+1}} \cdot \frac{k^2+1}{k^2+2k+2} = \frac{|x|}{3}.$$

So conv if $\frac{|x|}{3} < 1$, i.e. $-3 < x < 3$. Series conv. absolutely at endpts by p-test ($p=2$), so

**Answer:** Interval of Convergence = $\boxed{[-3, 3]}$

**b**. [3 points] Consider the power series $\sum_{j=0}^{\infty} C_j (x-2)^j$.

This power series converges when $x = -1$ and diverges when $x = 7$.
Which, if any, of the following intervals could be exactly equal to the interval of convergence for this power series? Circle all the intervals below that could be exactly equal to the interval of convergence or circle "NONE OF THESE".




$\Rightarrow 3 \le$ radius $\le 5$
centered at 2 $\Rightarrow$ endpts add to 4

$\boxed{[-2, 6]}$ $\quad$ ~~[−2, 7)~~ $\quad$ $\boxed{[-3, 7)}$ $\quad$ ~~(−3, 7]~~ $\quad$ ~~[−1, 4]~~ $\quad$ ~~[−4, 7)~~ $\quad$ NONE OF THESE