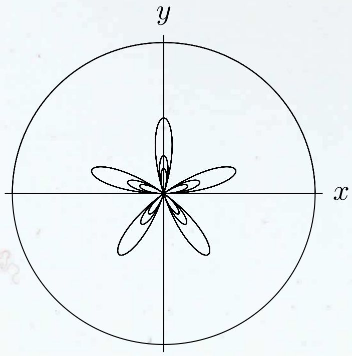
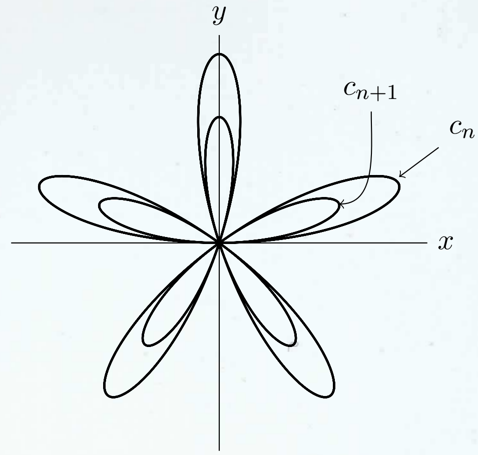
8. [8 points] A very unique sand dollar has an interesting pattern on it. The outline of the sand dollar is given by the polar equation $r = 2$. On the face of the sand dollar are many other polar curves c_n. For $n = 1, 2, 3, \ldots$, the curve c_n is given by the polar equation $r = \frac{1}{n}\sin(5\theta)$. Below is a picture of the sand dollar (left) and a zoomed in view of two of the polar curves (right).





a. [3 points] Let a_0 be the area that is inside the sand dollar but outside c_1. Write a formula involving one or more integrals for a_0.

$r = 0 \iff \sin 5\theta = 0 \iff 5\theta = k\pi$ for some integer k

$\iff \theta = k\pi/5$ for some integer k. So

$a_0 = 4\pi - 5(\text{area of one petal of } c_1) = 4\pi - 5 \cdot \frac{1}{2}\int_0^{\pi/5} \sin^2(5\theta)\,d\theta$

$$\boxed{= 4\pi - \frac{5}{2}\int_0^{\pi/5} \sin^2(5\theta)\,d\theta}$$

b. [3 points] For $n \geq 1$, let a_n be the area inside c_n but outside c_{n+1}. Write a formula involving one or more integrals for a_n when $n \geq 1$.

$$a_n = \boxed{\frac{5}{2}\int_0^{\pi/5} \frac{1}{n^2}\sin^2(5\theta)\,d\theta - \frac{5}{2}\int_0^{\pi/5} \frac{1}{(n+1)^2}\sin^2(5\theta)\,d\theta}$$

c. [2 points] Does the infinite series $\sum_{n=0}^{\infty} a_n$ converge or diverge? If it converges, what is its value? No justification is necessary.

$a_0 + a_1 + a_2 + \cdots$ converges to the area of the whole sand dollar, which is $\boxed{4\pi}$.