4. [11 points]

a. [6 points] Determine whether the following series converges absolutely, converges conditionally, or diverges, and give a complete argument justifying your answer.

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{1}{n}\right)$$

Converges absolutely $\boxed{\textbf{Converges conditionally}}$ **Diverges**

Justification:

Solution: This series converges by the alternating series test, which applies, since $\sin(\frac{1}{n})$ is a positive decreasing sequence that converges to zero.

It does not converge absolutely since for $n \geq 1$

$$\frac{1}{2n} \leq \sin\left(\frac{1}{n}\right).$$

We know the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges by p-test with $p = 1$. Then by the comparison test, so must $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty}\left|(-1)^n \sin\left(\frac{1}{n}\right)\right|$.

Alternatively, we can use the Limit Comparison Test. Since

$$\lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{x\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{y\to 0}\frac{\sin(y)}{y} = 1 < 0,$$

we know that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ must either both converge or both diverge. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series, which we know diverges, $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ must diverge as well.

b. [5 points] Compute the value of the following improper integral. **Show all your work using correct notation.** Evaluation of integrals must be done **without a calculator.**

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx$$

Solution:
First we change to limit notation, then use u-substitution with $u = 1 + e^x$.

$$\begin{aligned}\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx \\ &= \lim_{b\to\infty}\int_2^b \frac{1}{u^2}\,du \\ &= \lim_{b\to\infty} -\frac{1}{u}\Big|_2^b \\ &= \lim_{b\to\infty} \frac{-1}{b} - \frac{-1}{2} = \frac{1}{2}\end{aligned}$$

Alternatively, first compute the antiderivative using u-substitution.

$$\int \frac{e^x}{(1+e^x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{1+e^x}.$$

Thus,

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{1+e^b} + \frac{1}{2} = \frac{1}{2}.$$