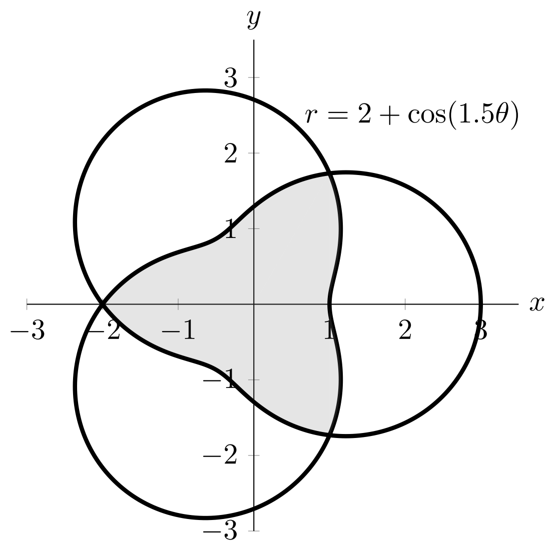
6. [11 points]

Leight Vloss has instructed his Star Children to run laps on the Trail of Atonement. The Trail of Atonement is best described as the polar curve $r = 2 + \cos(1.5\theta)$ where r is measured in kilometers. An aerial view of the trail is illustrated below.



a. [4 points] Leight stands on a pedestal in the center of the trail (at the origin). What is the furthest distance, in km, a Star Child gets from Leight on the Trail of Atonement? List all angles θ in $[0, 4\pi)$ where this distance r is achieved.

Solution: Since the maximum value of $\cos(1.5\theta)$ is 1 it follows that r is at most 3. To find the angles where this maximum r is achieved, we set

$$\begin{aligned} 2 + \cos(1.5\theta) &= 3 \\ \cos(1.5\theta) &= 1 \\ \frac{3}{2}\theta &= 0, 2\pi, 4\pi, \ldots \end{aligned}$$

So the values in $[0, 4\pi)$ are $\theta = 0, \frac{4\pi}{3}, \frac{8\pi}{3}$.

Alternatively, since we're looking for angles where r achieves its maximum, we can set the derivative of $r = 2 + \cos(1.5\theta)$ equal to 0. Since $r'(\theta) = -1.5\sin(1.5\theta)$ we have that $\theta = 0, \frac{4\pi}{3}, \frac{8\pi}{3}$.

Answer: Greatest distance: **3 km**

Answer: $\theta =$ $\theta = 0, \frac{4\pi}{3}, \frac{8\pi}{3}$

b. [3 points] Write an integral in terms of θ which represents the total length, in km, of the Trail of Atonement.

Solution: Since the Trail of Atonement is described by $r = f(\theta) = 2 + \cos(1.5\theta)$ with $0 \le \theta < 4\pi$ we use the polar arc length formula to write

$$\int_0^{4\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{4\pi} \sqrt{(2 + \cos(1.5\theta))^2 + (-1.5\sin(1.5\theta))^2}\, d\theta.$$

Alternatively, if we use $x = (2 + \cos(1.5\theta)\sin(\theta)$ and $y = (2 + \cos(1.5\theta))\sin(\theta)$, and then the parametric formula for arc length, we get

$$\int_0^{4\pi} \sqrt{\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta$$

where

$$\frac{dy}{d\theta} = -1.5\sin(1.5\theta)\sin(\theta) + (2 + \cos(1.5\theta))\cos(\theta)$$

and

$$\frac{dx}{d\theta} = -1.5\sin(1.5\theta)\cos(\theta) - (2 + \cos(1.5\theta))\sin(\theta)$$

c. [4 points] The shaded innermost region of the trail is called the Sacred Heart. Write an expression involving one or more integrals which represents the area, in km^2, of the Sacred Heart.

Solution: The corners of the Sacred heart occur when $r = 2$ as we can see from the leftmost corner. The first two solutions of

$$2 = 2 + \cos(1.5\theta)$$

are $\theta = \frac{\pi}{3}, \pi$. These angles describe 1/3 of the Sacred Heart, so the total area is

$$3\int_{\frac{\pi}{3}}^{\pi} \frac{f(\theta)^2}{2}\, d\theta = 3\int_{\frac{\pi}{3}}^{\pi} \frac{(2 + \cos(1.5\theta))^2}{2}\, d\theta.$$