8. [8 points] Let $f(x) = xe^{-x^2}$.

a. [4 points] Find the first four nonzero terms of the Taylor series for $f(x)$ centered at $x = 0$.

Answer: $x - x^3 + \frac{x^5}{2} - \frac{x^7}{6}$

b. [2 points] Find the value of $f^{(18)}(0)$.

Solution: Since $f(x)$ is an odd function, there are no odd powers of x in the Taylor series expansion of $f(x)$ centered at $x = 0$. Since the coefficient of x^{18} is $\frac{f^{(18)}(0)}{18!}$, it follows that $f^{(18)}(0) = 0$.

Answer: $f^{(18)}(0) =$ **0**

c. [2 points] Compute the limit

$$\lim_{x\to 0} \frac{xe^{-x^2} - x}{5x^3}.$$

Solution: Using the Taylor polynomial found in the first part we have

$$\lim_{x\to 0} \frac{xe^{-x^2} - x}{5x^3} = \lim_{x\to 0} \frac{-x^3 + \frac{x^5}{2} - \frac{x^7}{6}}{5x^3} = \lim_{x\to 0} -\frac{1}{5} + \frac{x^2}{10} - \frac{x^4}{30} = -\frac{1}{5}.$$

Note that this answer is exact and not an approximation, since all later terms in the Taylor series have x^n for $n > 7$, and so will go to 0 even when divided by x^3.

Answer: $\lim_{x\to 0} \frac{xe^{-x^2} - x}{5x^3} =$ $-\frac{1}{5}$