2. [8 points]

The castar, a coin widely used in Middle-Earth, allegedly has the shape graphed to the right. The outer perimeter can be modeled by the implicit equation

\[ x^4 + y^4 = 1 \]

and the perimeter of the hole in the middle is a square. To help his fellow Hobbits detect counterfeit coins, Samwise Gamgee, the Mayor of the Shire, is working on obtaining the specifications of a genuine castar. Sam needs your help.

a. [2 points] Find a function \( f(\theta) \) so that the outer edge of the castar is given by the function \( r = f(\theta) \).

\[
\begin{align*}
    x &= r \cos \theta, \quad y = r \sin \theta, \quad \text{so} \\
    (r \cos \theta)^4 + (r \sin \theta)^4 &= 1 \\
    \Rightarrow \quad r^4 \cos^4 \theta + r^4 \sin^4 \theta &= 1
\end{align*}
\]

**Answer:** \( f(\theta) = \left( \frac{1}{\cos^4 \theta + \sin^4 \theta} \right)^{1/4} \)

b. [3 points] Write an expression involving one or more integrals that gives the total area of the quarter of a castar in the first quadrant (shaded above).

Area of shaded triangle = \( \frac{1}{2} \), so

Area of quarter coin = \( \frac{1}{2} \int_0^{\pi/2} f(\theta)^2 \, d\theta \cdot \frac{1}{2} \)

**Answer:** \( \frac{1}{2} \int_0^{\pi/2} \left( \frac{1}{\cos^4 \theta + \sin^4 \theta} \right)^{1/2} \, d\theta - \frac{1}{2} \)

c. [3 points] Approximate the area of a castar by estimating your integral(s) from part (b) using TRAP(2). Write out all the terms in your sum(s).

For \( \int_0^B g(x) \, dx \), \( \text{TRAP}(2) = \Delta x \left[ \frac{1}{2} g(a) + g \left( \frac{B}{2} \right) + \frac{1}{2} g(b) \right] \), where \( \Delta x = \frac{B}{4} \).

In our case \( B = \frac{\pi}{2} \) so \( \frac{B}{2} = \Delta x = \frac{\pi}{4} \).

\[
\int_0^{\pi/2} f(\theta)^2 \, d\theta = 4 \left( \frac{\pi}{8} \left( \frac{1}{1 + \sqrt{2}} \right) - \frac{1}{2} \right) = \frac{\pi}{2} (1 + \sqrt{2}) - 2 \approx 1.79
\]