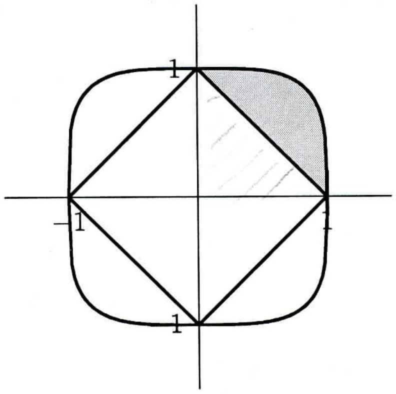
2. [8 points]

The *castar*, a coin widely used in Middle-Earth, allegedly has the shape graphed to the right. The outer perimeter can be modeled by the implicit equation

$$x^4 + y^4 = 1$$

and the perimeter of the hole in the middle is a square. To help his fellow Hobbits detect counterfeit coins, Samwise Gamgee, the Mayor of the Shire, is working on obtaining the specifications of a genuine castar. Sam needs your help.



a. [2 points] Find a function $f(\theta)$ so that the outer edge of the castar is given by the function $r = f(\theta)$.

$x = r\cos\theta$, $y = r\sin\theta$, so

$(r\cos\theta)^4 + (r\sin\theta)^4 = 1$

$\Rightarrow r^4\cos^4\theta + r^4\sin^4\theta = 1$

Answer: $f(\theta) = \left(\dfrac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/4}$

b. [3 points] Write an expression involving one or more integrals that gives the total area of the quarter of a castar in the first quadrant (shaded above).

Area of shaded triangle $= \frac{1}{2}$, so

area of quarter coin $= \frac{1}{2}\int_0^{\pi/2} f(\theta)^2\, d\theta - \frac{1}{2}$

Answer: $\frac{1}{2}\int_0^{\pi/2}\left(\dfrac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/2} d\theta - \frac{1}{2}$

c. [3 points] Approximate the area of a castar by estimating your integral(s) from part (b) using TRAP(2). Write out all the terms in your sum(s).

For $\int_0^B g(x)\,dx$, $\text{TRAP}(2) = \Delta x\left[\frac{1}{2}g(0) + g(\frac{B}{2}) + \frac{1}{2}g(B)\right]$, where $\Delta x = \frac{B}{2}$.

In our case $B = \frac{\pi}{2}$ so $\frac{B}{2} = \Delta x = \frac{\pi}{4}$.

four quarters

Answer: $4\left[\frac{1}{2}\cdot\frac{\pi}{4}\left[\frac{1}{2}\left(\dfrac{1}{\cos^4(0)+\sin^4(0)}\right)^{1/2} + \left(\dfrac{1}{\cos^4(\frac{\pi}{4})+\sin^4(\frac{\pi}{4})}\right)^{1/2} + \frac{1}{2}\left(\dfrac{1}{\cos^4(\frac{\pi}{2})+\sin^4(\frac{\pi}{2})}\right)^{1/2}\right] - 1/2\right]$

$= 4\left[\frac{\pi}{8}(1+\sqrt{2}) - \frac{1}{2}\right] = \frac{\pi}{2}(1+\sqrt{2}) - 2 \approx 1.79$