6. [11 points] Consider the function $g(x)$ defined for all real numbers represented by the Taylor series

$$-2^{2019}\frac{2020!}{1011!}$$

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n-1}}{(n+1)!} x^{2n}.$$

a. [3 points] Find the values of $g^{(2019)}(0)$ and $g^{(2020)}(0)$. You do *not* need to simplify.

Answer: $g^{(2019)}(0) =$ 0 $g^{(2020)}(0) =$ ______

$g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k$. Since all terms in the sum have even degree, $g^{(2019)}(0) = 0$. There is an x^{2020} term, when $n = 1010$.

So $\frac{g^{(2020)}(0)}{2020!} = (-1)^{1010-1} \frac{2^{2020-1}}{(1010+1)!}$

b. [2 points] Find $P_4(x)$, the Taylor polynomial of $g(x)$ of degree 4 near $x = 0$.

$n=1 : (-1)^{1-1} \frac{2^{2\cdot 1-1}}{(1+1)!} x^{2\cdot 1} = x^2$

$n=2 : (-1)^{2-1} \frac{2^{2\cdot 2-1}}{(2+1)!} x^{2\cdot 2} = -\frac{8}{6} x^4$

Answer: $P_4(x) =$ $x^2 - \frac{4}{3}x^4$

c. [3 points] Define

$$G(x) = \int_{-1}^{x} g(t)\, dt.$$

Use $P_4(x)$ to estimate $G(2)$.

Answer: $G(2) \approx$ -5.8

$G(2) = \int_{-1}^{2} g(t)\,dt \approx \int_{-1}^{2} P_4(t)\,dt = \int_{-1}^{2} t^2 - \frac{4}{3}t^4\,dt = \frac{1}{3}t^3 - \frac{4}{15}t^5 \Big|_{-1}^{2}$

$= \left[\frac{1}{3}(2)^3 - \frac{4}{15}(2)^5\right] - \left[\frac{1}{3}(-1)^3 - \frac{4}{15}(-1)^5\right] = \frac{8}{3} - \frac{128}{15} + \frac{1}{3} - \frac{4}{15}$

d. [3 points] Use an appropriate Taylor polynomial to compute the limit

$$\lim_{x \to 0^+} \frac{g'(x)}{x}$$

Show your work carefully.

Answer: 2

$g(x) = x^2 +$ some terms of degree at least 4

so $g'(x) = 2x +$ " " " " " 3

so $\frac{g'(x)}{x} = 2 + \underbrace{\text{" " " " " 2}}_{\text{these} \to 0 \text{ as } x \to 0}$