7. [13 points] The parts of this problem are unrelated.

a. [3 points] Consider the function

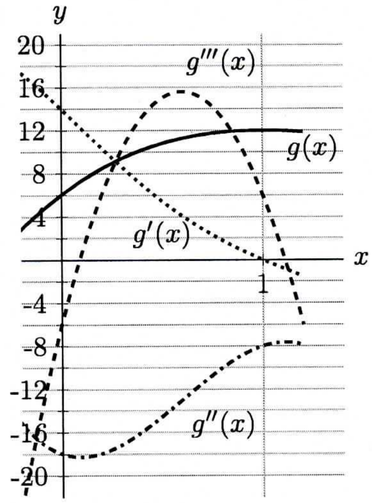
$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{\sin x}{x} - \cos x & \text{for } x \neq 0 \end{cases}$$

Find the Taylor series for $f(x)$ centered at $x = 0$. Write your answer as a single sum using sigma notation.

$$f(x) = \frac{1}{x}\sin(x) - \cos(x) = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n+1)!} - \frac{1}{(2n)!}\right] x^{2n}$$

Answer: $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \left[\frac{-2n}{(2n+1)!}\right] x^{2n}$

b. [4 points] Part of the graphs of $g(x), g'(x), g''(x)$, and $g'''(x)$ are given to the right.
Find the third-degree Taylor polynomial for $g(x)$ near $x = 1$.



$g(1) = 12$
$g'(1) = 0$
$g''(1) = -8$
$g'''(1) = 6$

$$P_3(x) = g(1) + g'(1)(x-1) + \tfrac{1}{2}g''(1)(x-1)^2 + \tfrac{1}{6}g'''(1)(x-1)^3$$

$$= 12 + (0)(x-1) + \tfrac{1}{2}(-8)(x-1)^2 + \tfrac{1}{6}(6)(x-1)^3$$

Answer: $12 - 4(x-1)^2 + (x-1)^3$

c. [6 points] Find the exact value (in closed form) of the following series. You do not need to justify your answers.

i. $0.1 + \dfrac{0.01}{2} + \dfrac{0.001}{3} + \dfrac{0.0001}{4} + \cdots =$ $\quad -\ln(.9) = \ln 10 - \ln 9$

ii. $\dfrac{\pi}{2} - \dfrac{3}{\pi} + \dfrac{18}{\pi^3} - \dfrac{108}{\pi^5} + \cdots = \dfrac{\pi}{2}\left[1 - \dfrac{6}{\pi^2} + \dfrac{6^2}{\pi^4} - \dfrac{6^3}{\pi^6} + \cdots\right]$ $\quad \pi^3/(2\pi^2+12)$

iii. $\dfrac{1}{2} - 2e^2 + \dfrac{2^3e^4}{3!} - \dfrac{2^5e^6}{5!} + \cdots = \dfrac{1}{2} - e\left[2e - \dfrac{2^3e^3}{3!} + \dfrac{2^5e^5}{5!} - \cdots\right]$ $\quad \dfrac{1}{2} - e\sin(2e)$

i) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

So $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$

So $-\ln(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$

So $-\ln(1-.1) = .1 + \dfrac{.01}{2} + \dfrac{.001}{3} + \dfrac{.0001}{4} + \cdots$

ii) is geometric with $a = \dfrac{\pi}{2}$, $x = \dfrac{-6}{\pi^2}$

so $\dfrac{\pi/2}{1 - \frac{-6}{\pi^2}} \cdot \dfrac{2\pi^2}{2\pi^2} = \dfrac{\pi^3}{2\pi^2+12}$

iii) Looks like sin:

$\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$