11. [12 points] Let $f(x) = x(1-x)^{-1/2}$.

a. [4 points] Write down the first 3 non-zero terms of the Taylor series for $f(x)$ centered at $x = 0$. Show your work.

Solution: From the list of "Known" Taylor series:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots$$

Put $p = -\frac{1}{2}$:

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Replace x by $-x$:

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Multiply by x:

$$x(1-x)^{-1/2} = x + \frac{1}{2}x^2 + \frac{3}{8}x^3 + \cdots$$

b. [3 points] Let $F(x)$ be an antiderivative of $f(x)$ such that $F(0) = 2$. Write down the first 4 non-zero terms of the Taylor series for $F(x)$ centered at $x = 0$. Show your work.

Solution: By the second fundamental theorem of calculus,

$$F(x) = 2 + \int_0^x f(t)\, dt.$$

Therefore,

$$F(x) = 2 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \cdots$$

c. [5 points] Compute the exact value of $\displaystyle\int_0^{3/4} f(x)\, dx$. Show each step of your computation.

Solution: Integration by parts: Let $u = x$ and $v' = (1-x)^{-1/2}$. Then $u' = 1$ and $v = -2(1-x)^{1/2}$. We have

$$\begin{aligned}\int_0^{3/4} x(1-x)^{-1/2}\, dx &= -2x(1-x)^{1/2}\Big|_0^{3/4} + 2\int_0^{3/4} (1-x)^{1/2}\, dx \\ &= \left(-2x(1-x)^{1/2} - \frac{4}{3}(1-x)^{3/2}\right)_0^{3/4} \\ &= \left(-\frac{3}{4} - \frac{1}{6}\right) + \frac{4}{3} = \frac{5}{12}.\end{aligned}$$