4. [12 points] Let

$$G(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n5^n}x^n.$$

a. [6 points] What is the **interval** of convergence for $G(x)$? Show the mechanics of any tests or theorems you use. Take as given, and **do not show**, that the radius of convergence of $G(x)$ is 5.

Solution: The center is at $x=0$, so the two end points are 5 and -5.
At $x=5$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n5^n}5^n=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

Since $\frac{1}{n}$ is positive, decreasing, and $\lim_{n\to\infty}\frac{1}{n}=0$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by alternating series test.
At $x=-5$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n5^n}(-5)^n=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n5^n}(-1)^n5^n=-\sum_{n=1}^{\infty}\frac{1}{n},$$

which diverges by p-test, $p=1$.
Therefore, the interval of convergence is $(-5,5]$.

b. [3 points] Find $G^{(100)}(0)$.

Solution: Consider the coefficient of x^{100} of the power series.

$$\frac{G^{(100)}(0)}{100!}=\frac{(-1)^{101}}{100\cdot 5^{100}},$$

$$G^{(100)}(0)=-\frac{100!}{100\cdot 5^{100}}.$$

c. [3 points] Compute the exact value of $G(2)$.

Solution:

$$G(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n5^n}2^n=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{2}{5}\right)^n.$$

From the list of "Known" Taylor series,

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$$

for $-1<x\le 1$. Plug in $x=2/5$, we have that $G(2)=\ln(1+\frac{2}{5})=\ln(7/5)$.