4. [12 points] Let
\[ G(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} x^n. \]

a. [6 points] What is the interval of convergence for \( G(x) \)? Show the mechanics of any tests or theorems you use. Take as given, and do not show, that the radius of convergence of \( G(x) \) is 5.

**Solution:**
The center is at \( x = 0 \), so the two end points are 5 and -5.
At \( x = 5 \),
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} 5^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}. \]
Since \( \frac{1}{n} \) is positive, decreasing, and \( \lim_{n \to \infty} \frac{1}{n} = 0 \), the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \) converges by alternating series test.
At \( x = -5 \),
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} (-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} (-1)^n 5^n = -\sum_{n=1}^{\infty} \frac{1}{n}, \]
which diverges by \( p \)-test, \( p = 1 \).
Therefore, the interval of convergence is \((-5, 5)\).

b. [3 points] Find \( G^{(100)}(0) \).

**Solution:** Consider the coefficient of \( x^{100} \) of the power series.
\[ G^{(100)}(0) = \frac{(-1)^{101}}{100!} \cdot \frac{1}{5^{100}}, \]
\[ G^{(100)}(0) = -\frac{100!}{100 \cdot 5^{100}}. \]

c. [3 points] Compute the exact value of \( G(2) \).

**Solution:**
\[ G(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} 2^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{2}{5} \right)^n. \]

From the list of “Known” Taylor series,
\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \]
for \(-1 < x \leq 1\). Plug in \( x = 2/5 \), we have that \( G(2) = \ln(1 + \frac{2}{5}) = \ln(7/5) \).