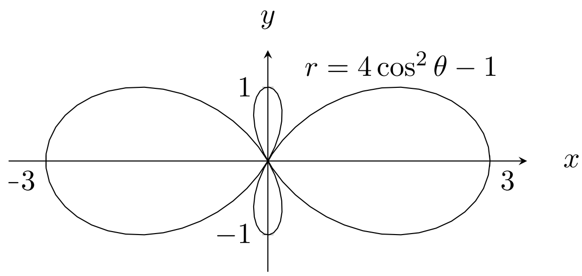
7. [15 points] Nat is sailing a boat in a lake, with the path given by the following polar graph.



a. [4 points] What are all the angles θ, with $0 \leq \theta \leq 2\pi$, for which the graph passes through the origin?

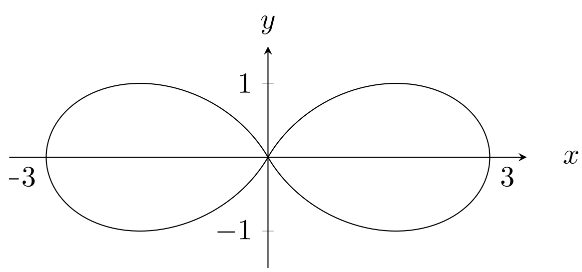
Solution: At the origin, $r = 0$.

$$4\cos^2\theta - 1 = 0$$
$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \frac{1}{2} \qquad \text{or} \qquad \cos\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \qquad \text{or} \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}.$$

So $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

b. [4 points] Write down, but do **not** evaluate, one or more integral(s) that gives the arc length of the larger horizontal figure 8 from the path above, as given in the following graph.



Solution: When $\theta = 0$, we have $r = 4 - 1 = 3$, so the x, y-coordinates of the point is $(3, 0)$. When θ increases, the graph travels counter-clockwise. The part in the 1st quadrant is traced when θ increases from $\theta = 0$ til the first time the graph meets the origin, i.e. $\pi/3$. As for the 4th quadrant, the portion of the graph is traced when θ decreases from $\theta = 0$ til the first time the graph meets the origin at a negative angle, i.e. $\theta = 5\pi/3 - 2\pi = -\pi/3$. Therefore, the right loop has arc length

$$\int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

By symmetry, the total arc length of the horizontal figure 8 is given by doubling the right loop, i.e.

$$2\int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

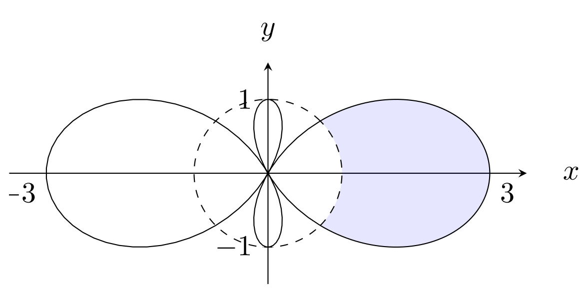
Alternatively, the left loop is traced from $\theta = 2\pi/3$ to $\theta = 4\pi/3$, so the total arc length can also be written as

$$\begin{aligned}\text{left loop} + \text{right loop} &= \int_{2\pi/3}^{4\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta \\ &\qquad + \int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.\end{aligned}$$

Alternatively, the total arc length of the horizontal figure 8 is 4 times the arc length of the portion in the first quadrant, by symmetry, so it is

$$4\int_{0}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

c. [5 points] Another boat is travelling around the unit circle $r = 1$, given by the dashed curve in the graph below. Write down, but do **not** evaluate, one or more integral(s) that gives the area of the shaded region, as shown below.



Solution: The two graphs intersect once at $0 < \theta < \pi/3$, and another time at $-\pi/3 < \theta < 0$. To solve for the angle of intersection, we set the two r to be equal.

$$4\cos^2\theta - 1 = 1$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \qquad \text{or} \qquad \cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}, \frac{7\pi}{4} \qquad \text{or} \qquad \theta = \frac{3\pi}{4}, \frac{5\pi}{4},$$

or adding integer multiples of 2π to any of these.
We want an angle in $0 < \theta < \dfrac{\pi}{3}$ and another in $-\dfrac{\pi}{3} < \theta < 0$, so we have $\theta = \dfrac{\pi}{4}$ and $-\dfrac{\pi}{4}$.
Therefore, the area bounded by the solid curve is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (4\cos^2\theta - 1)^2 \, d\theta.$$

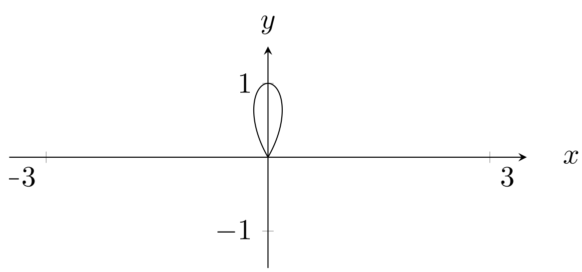
The area bounded by the dashed curve is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} 1^2 \, d\theta = \frac{\pi}{4},$$

so the shaded area is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (4\cos^2\theta - 1)^2 - 1^2 \, d\theta.$$

d. [2 points] Give an interval of θ-values for which the polar equation $r = 4\cos^2\theta - 1$ traces out the upper loop of the smaller figure 8 as shown below.



Solution: From the analysis in (b), we know that as θ increases from 0 to $\pi/3$, the first quadrant portion of the horizontal figure 8 is traced out.
Then as θ increases from $\pi/3$ to $2\pi/3$, we have that $r = 4\cos^2\theta - 1$ is negative. As a result, the direction of the point is **opposite** to the direction indicated by the angle θ. This means that we are below the x-axis instead of above the x-axis. As a result, the **lower** half of the small loop is traced out in $[\pi/3, 2\pi/3]$.
When θ increase from $2\pi/3$ to $4\pi/3$, the formula $r = 4\cos^2\theta - 1$ stays positive. Hence the direction is given by the angle indicated by the angle θ. At this interval, the left half of the big horizontal figure 8 is traced out.
Finally, when θ increases from $4\pi/2$ to $5\pi/3$, we have that $r = 4\cos^2\theta - 1$ is negative. Hence we need to go to the opposite direction as indicated by θ. As a result, we are at the **upper** half of the small loop when θ is in $[4\pi/3, 5\pi/3]$.