6. [9 points] Suppose the Taylor series for a function $f(x)$ around $x = 3$ is

$$\sum_{n=1}^{\infty} \frac{(6)^{-n}((2n)!)}{n!(n-1)!}(x-3)^{2n}$$

a. [6 points] Compute the radius of convergence for this series. Be sure to fully justify your answer and show all work. Do not compute the interval of convergence.

Solution: We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(6)^{-(n+1)}((2n+2)!)}{(n+1)!n!}\frac{n!(n-1)!}{(6)^{-(n)}((2n)!)}\frac{(x-3)^{2n+2}}{(x-3)^{2n}}\right| < 1$$

Pairing terms to cancel we get:

$$\lim_{n\to\infty}\left|\frac{(6)^{-(n+1)}}{(6)^{-(n)}}\frac{((2n+2)!)}{((2n)!)}\frac{n!(n-1)!}{(n+1)!n!}(x-3)^2\right| < 1$$

Then simplifying we are left with:

$$\lim_{n\to\infty}\left|\frac{1}{6}\cdot\frac{(2n+1)(2n+1)}{(n+1)n}\cdot(x-3)^2\right| < 1$$

Then, if we take the limit, by L'Hopital or a dominating functions argument, the radius, we get:

$$\frac{2}{3}|x-3|^2 < 1$$

which means $|x-3| < \left(\frac{3}{2}\right)^{1/2}$.

Radius of Convergence: $\sqrt{\frac{3}{2}}$

b. [3 points] Find $f^{(2022)}(3)$ and $f^{(2023)}(3)$.

Solution: A power series is its own Taylor series, so we get we just need to find n such that :

$$\frac{f^{(2022)}(3)}{(2022!)}(x-3)^{2022} = \frac{(6)^{-n}((2n)!)}{n!(n-1)!}(x-3)^{2n}$$

Comparing powers of $(x-3)$, we see that $2n = 2022$, and so $n = 1011$. This means:

$$\frac{f^{(2022)}(3)}{(2022!)} = \frac{(6)^{-1011}((2022)!)}{(1011!)(1010!)}$$

and so

$$f^{(2022)}(3) = \frac{(6)^{-1011}(2022!)^2}{(1011!)(1010!)}$$

Since the series is even, $f^{(2023)}(3) = 0$

$f^{(2022)}(3)=$ $\frac{(6)^{-1011}(2022!)^2}{(1011!)(1010!)}$ $f^{(2023)}(3)=$ 0