7. [6 points] Compute the interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(x-4)^n$$

Show all work and give all necessary justification. You may assume it has radius of convergence 4 and you do not need to show this.

Solution: We have that the radius of convergence is 4. We know that the series is centered at $x = 4$. Therefore, to compute the interval of convergence, we just need to check convergence at the endpoints, which occur at 0 and 8. At $x = 0$, the series becomes:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(0-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This diverges by p-test $p = 1$, so the series does not converge at $x = 0$, meaning the interval is now $(0, 8?$.
If we plug in $x = 8$, we get:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(8-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n(4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

As the terms $a_n = \frac{1}{n}$ are decreasing and going to zero, this series converges by the alternating series test. Since the series converges at $x = 8$, we include it in the radius of convergence

Interval of Convergence: $(0, 8]$