8. [11 points]
   a. [6 points] Give the first three non-zero terms of the Taylor Series for the function:

\[
(x^2 + 2) \sin(x)
\]

centered at \( x = 0 \).

Solution: The function \((x^2 + 2)\) is a polynomial, and therefore its own Taylor series. Next, we take the known Taylor series for \(\sin(x)\). Since we need the first three non-zero terms, we start with the first 3 non-zero terms of \(\sin(x)\). This gives:

\[
(x^2 + 2) \sin(x) \approx (x^2 + 2) \left( x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 \right)
\]

If we expand the right side, we get:

\[
x^3 - \frac{1}{3!} x^5 + \frac{1}{5!} x^7 + 2x - \frac{2}{3!} x^3 + \frac{2}{5!} x^5
\]

Now, if we group terms, we get:

\[
2x + (1 - \frac{2}{3!})x^3 + (\frac{2}{5!} - \frac{2}{3!}) x^5 + \frac{1}{5!} x^7
\]

Nothing has cancelled out, meaning the first three terms here are the first three non-zero terms we need. Therefore the final answer is:

\[
2x + \left( 1 - \frac{2}{3!} \right) x^3 + \left( \frac{2}{5!} - \frac{2}{3!} \right) x^5
\]
b. [5 points] Compute the limit:
\[
\lim_{x \to 0} \frac{\int_0^{2x} \left( \left( \frac{t}{2} \right)^2 + 2 \right) \sin \left( \frac{t}{2} \right) dt}{x^2}
\]

**Solution:** Solutions 1 and 2): If we examine the bounds of the integral, we see that as \( x \to 0 \), the numerator becomes zero. This means that the answer is in indeterminant form, and we can apply L'Hopital. Using second FTC, and chain rule, we get that the limit becomes:
\[
\lim_{x \to 0} \frac{2((x^2 + 2) \sin(x))}{2x} = \lim_{x \to 0} \frac{(x^2 + 2) \sin(x)}{x}
\]
From here, the limit becomes \( \frac{0}{0} \) again, so we have two options. We can apply L'Hopital again, turning the limit into:
\[
\lim_{x \to 0} \frac{(2x \sin(x) + (x^2 + 2) \cos(x))}{1} = \frac{(0 + (0^2 + 2) \cos(0))}{1} = 2
\]
Or, instead of L'Hopital, we can plug in the answer from a) to get:
\[
\lim_{x \to 0} \frac{2x + \left( 1 - \frac{2}{3!} \right) x^3 + \left( \frac{2}{5} - \frac{2}{3!} \right) x^5}{x} = \lim_{x \to 0} 2 + \left( 1 - \frac{2}{3!} \right) x^2 + \left( \frac{2}{5!} - \frac{2}{3!} \right) x^4 = 2 + 0 + 0 = 2.
\]
Solution 3): It is also possible to integrate the answer from a). Doing a u-sub, with \( u = t/2 \), before taking the limit, we get
\[
\lim_{x \to \infty} \frac{2 \int_0^x (u^2 + 2) \sin(u) du}{x^2}
\]
Subbing in the answer from a) we get:
\[
\lim_{x \to \infty} \frac{2 \int_0^x 2u + \left( 1 - \frac{2}{3!} \right) u^3 + \left( \frac{2}{5} - \frac{2}{3!} \right) u^5 du}{x^2}
\]
This becomes:
\[
\lim_{x \to \infty} \frac{2(2x^2 + 1 \cdot 4 \left( 1 - \frac{2}{3!} \right) x^4 + 1 \cdot \frac{1}{6} \left( \frac{2}{5!} - \frac{2}{3!} \right) x^6)}{x^2}
\]
Which becomes:
\[
\lim_{x \to \infty} 2 \left( 1 + \frac{4}{4} \left( 1 - \frac{2}{3!} \right) x^2 + \frac{1}{6} \left( \frac{2}{5!} - \frac{2}{3!} \right) x^4 \right) = 2(1 + 0 + 0) = 2
\]
Answer: 2