8. [11 points]

a. [6 points] Give the first three non-zero terms of the Taylor Series for the function:

$$(x^2+2)\sin(x)$$

centered at $x=0$.

Solution: The function (x^2+2) is a polynomial, and therefore its own Taylor series. Next, we take the known Taylor series for $\sin(x)$. Since we need the first three non-zero terms, we start with the first 3 non-zero terms of $\sin(x)$. This gives:

$$(x^2+2)\sin(x) \approx (x^2+2)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5\right)$$

If we expand the right side, we get:

$$x^3 - \frac{1}{3!}x^5 + \frac{1}{5!}x^7 + 2x - \frac{2}{3!}x^3 + \frac{2}{5!}x^5$$

Now, if we group terms, we get:

$$2x + (1 - \frac{2}{3!})x^3 + (\frac{2}{5!} - \frac{2}{3!})x^5 + \frac{1}{5!}x^7$$

Nothing has cancelled out, meaning the first three terms here are the first three non-zero terms we need. Therefore the final answer is:

$$2x + \left(1 - \frac{2}{3!}\right)x^3 + \left(\frac{2}{5!} - \frac{2}{3!}\right)x^5$$

b. [5 points] Compute the limit:

$$\lim_{x\to 0} \frac{\int_0^{2x} \left(\left(\frac{t}{2}\right)^2 + 2\right) \sin\left(\frac{t}{2}\right) dt}{x^2}$$

Solution: Solutions 1 and 2): If we examine the bounds of the integral, we see that as $x \to 0$, the numerator becomes zero. This means that the answer is in indeterminant form, and we can apply L'Hopital. Using second FTC, and chain rule, we get that the limit becomes:

$$\lim_{x\to 0} \frac{2((x^2+2)\sin(x))}{2x} = \lim_{x\to 0} \frac{(x^2+2)\sin(x)}{x}$$

From here, the limit becomes $\frac{0}{0}$ again, so we have two options. We can apply L'Hopital again, turning the limit into:

$$\lim_{x\to 0} \frac{(2x\sin(x) + (x^2+2)\cos(x))}{1} = \frac{(0 + (0^2+2)\cos(0))}{1} = 2$$

Or, instead of L'Hopital, we can plug in the answer from a) to get:

$$\lim_{x\to 0} \frac{2x + \left(1 - \frac{2}{3!}\right)x^3 + \left(\frac{2}{5!} - \frac{2}{3!}\right)x^5}{x} = \lim_{x\to 0} 2 + \left(1 - \frac{2}{3!}\right)x^2 + \left(\frac{2}{5!} - \frac{2}{3!}\right)x^4 = 2+0+0 = 2.$$

Solution 3): It is also possible to integrate the answer from a). Doing a u-sub, with $u = t/2$, before taking the limit, we get

$$\lim_{x\to\infty} \frac{2\int_0^x (u^2+2)\sin(u)du}{x^2}$$

Subbing in the answer from a) we get:

$$\lim_{x\to\infty} \frac{2\int_0^x 2u + \left(1 - \frac{2}{3!}\right)u^3 + \left(\frac{2}{5!} - \frac{2}{3!}\right)u^5)du}{x^2}$$

This becomes:

$$\lim_{x\to\infty} \frac{2(x^2 + \frac{1}{4}\left(1 - \frac{2}{3!}\right)x^4 + \frac{1}{6}\left(\frac{2}{5!} - \frac{2}{3!}\right)x^6)}{x^2}$$

Which becomes:

$$\lim_{x\to\infty} 2(1 + \frac{1}{4}\left(1 - \frac{2}{3!}\right)x^2 + \frac{1}{6}\left(\frac{2}{5!} - \frac{2}{3!}\right)x^4) = 2(1+0+0) = 2$$

Answer: 2