10. [10 points] For each of the following parts, circle **all** correct answers. No justification is required.

a. [2 points] A power series $\sum_{n=0}^{\infty} A_n(x+1)^n$ **converges** at $x=-3$ and **diverges** at $x=2$.

At which of the following x-values, if any, **must** this power series **converge?**

$x=-4$ $\boxed{x=-2}$ $\boxed{x=0}$ $x=1$ $x=3$ NONE

Solution: The center of convergence is $x=-1$. Since the power series converges at $x=-3$, its radius of convergence must be at least 2, so its interval of convergence must contain the interval $[-3,1)$. So the power series **must** converge at $x=-2$ and $x=0$. Of the x-values listed, there are no others where the power series **must** converge, because $[-3,1)$ could be the exact interval of convergence of this power series.

b. [2 points] Another power series $\sum_{n=0}^{\infty} B_n(x+1)^n$ **converges** for all $x<-3$.

At which of the following x-values, if any, **must** this power series **converge?**

$\boxed{x=-4}$ $\boxed{x=-2}$ $\boxed{x=0}$ $\boxed{x=1}$ $\boxed{x=3}$ NONE

Solution: If a power series converges for all $x<-3$, then its interval of convergence must contain the interval $(-\infty,-3)$. But then the radius of convergence must be ∞, so the power series must in fact converge for all $x\geq -3$, too.

c. [2 points] Which, if any, of the following infinite series **converge** to $\frac{1}{2}$?

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ | $\boxed{\sum_{n=0}^{\infty} \frac{3}{8}\left(\frac{1}{4}\right)^n}$ | $\sum_{n=1}^{\infty} \frac{n^2+2}{2n^2}$ | $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}}$ | NONE

Solution: First, using the "known" Taylor series for $\ln(1+x)$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot 1^n = \ln(1+1) = \ln(2) \neq \frac{1}{2}.$$

Next, using the formula for the sum of an infinite geometric series, we have

$$\sum_{n=0}^{\infty} \frac{3}{8}\left(\frac{1}{4}\right)^n = \frac{\frac{3}{8}}{1-\frac{1}{4}} = \frac{\frac{3}{8}}{\frac{3}{4}} = \frac{3}{8} \cdot \frac{4}{3} = \frac{1}{2}.$$

Now, note that $\lim_{n\to\infty} \frac{n^2+2}{2n^2} = \frac{1}{2} \neq 0$. By the ***n*th term test for divergence,** the series $\sum_{n=1}^{\infty} \frac{n^2+2}{2n^2}$ is **divergent** (so it cannot equal $\frac{1}{2}$).

Finally, using the "known" Taylor series for $\cos(x)$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

d. [2 points] Consider the curve traced out by the parametric equations:

$$\big(x(t),\ y(t)\big) = \big(t^2,\ \sin(\pi t)\big) \qquad \text{for } t \geq 0.$$

Which, if any, of the following is the **slope** of the tangent line to this curve at $t = 1$?

$\frac{2}{\pi}$ | $-\frac{2}{\pi}$ | $\frac{\pi}{2}$ | $\boxed{-\frac{\pi}{2}}$ | 2π | -2π | NONE

Solution: The slope is given by $\frac{dy/dt}{dx/dt}$ at $t = 1$. We have $x'(t) = 2t$ and $y'(t) = \pi\cos(\pi t)$, so $\left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \frac{y'(1)}{x'(1)} = -\frac{\pi}{2}$.

e. [2 points] Which, if any, of the following points given in **polar coordinates (r, θ)** represent the same point as $(x, y) = (-1, 0)$ in the xy-plane?

$\boxed{(r,\theta) = (1, 3\pi)}$ | $\boxed{(r,\theta) = (1, \pi)}$ | $(r,\theta) = (-1, \pi)$ | $\boxed{(r,\theta) = (-1, 0)}$ | NONE

Solution: Use the formulas $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to convert each listed point from polar coordinates to Cartesian coordinates. All of them are converted to the point $(x, y) = (-1, 0)$, except for the point $(r, \theta) = (-1, \pi)$, which is converted to the point $(x, y) = (1, 0)$.