3. [9 points] The Taylor series centered at $x = 1$ for a function $T(x)$ is given by:

$$T(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(-5)^n \cdot (2n)!} (x-1)^{4n+3}.$$

a. [6 points] Find the **radius of convergence** of the Taylor series above. Show your work. Do not attempt to find the interval of convergence.

Solution: We use the ratio test: letting $a_n = \dfrac{(n!)^2(x-1)^{4n+3}}{(-5)^n \cdot (2n)!}$, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{((n+1)!)^2(x-1)^{4n+7}}{(-5)^{n+1}\cdot(2n+2)!}\cdot\frac{(-5)^n\cdot(2n)!}{(n!)^2(x-1)^{4n+3}}\right| \\ &= \lim_{n\to\infty}\frac{(n+1)(n+1)|x-1|^4}{5(2n+2)(2n+1)} \\ &= \frac{|x-1|^4}{20}\end{aligned}$$

The series converges when $\dfrac{|x-1|^4}{20} < 1$; that is, when $|x-1|^4 < 20$, and so $|x-1| < 20^{1/4}$. Therefore the radius of convergence is $20^{1/4}$.

Answer: $20^{1/4}$

b. [3 points] Compute $T^{(123)}(1)$. Show your work. You do not need to simplify your answer.

Solution: The quantity $\dfrac{T^{(123)}(1)}{123!}$ is the coefficient of $(x-1)^{123}$ in the Taylor series for $T(x)$. Using the formula for the Taylor series of $T(x)$, we must find n such that

$$\frac{T^{(123)}(1)}{123!}(x-1)^{123} = \frac{(n!)^2}{(-5)^n\cdot(2n)!}(x-1)^{4n+3}.$$

Comparing powers of $x-1$, we solve $123 = 4n+3$ to get $n = 30$, and so we obtain

$$\frac{T^{(123)}(1)}{123!} = \frac{(30!)^2}{(-5)^{30}\cdot 60!}.$$

Now we multiply both sides by 123! to get the answer.

Answer: $T^{(123)}(1) = \dfrac{(30!)^2\cdot 123!}{(-5)^{30}\cdot 60!}$