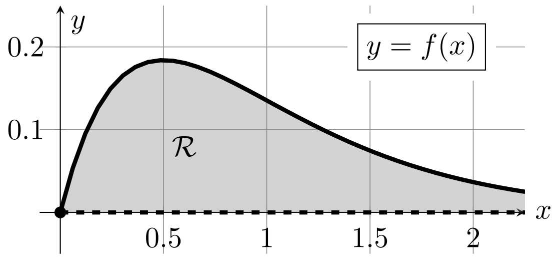
4. [10 points] Louise, the world-famous abstract artist and cheese enthusiast, has a dream about an infinitely-long cheese sculpture. The sculpture involves $\mathcal{R}$, which is the region above the x-axis and below the curve $f(x) = xe^{-2x}$ on the interval $[0, \infty)$. A portion of $\mathcal{R}$ is the shaded region below.



a. [4 points] Write an improper integral that represents the **volume** of the infinitely-long solid of revolution formed by rotating the region $\mathcal{R}$ around the x-axis.
Your answer should not involve the letter f. **Do not evaluate your integral.**

Solution: We use the disk method (the washer method but with no inner radius). The disk method states that the volume of a disk of thickness Δx at a point $x \geq 0$ is of the form $\pi R(x)^2 \Delta x$ for some function $R(x)$ representing the radius of our disk. For each $x \geq 0$, the radius $R(x)$ is given by the distance from the x-axis to $y = f(x)$, which is $R(x) = xe^{-2x}$. This gives us our answer:

$$\int_0^\infty \pi \left(xe^{-2x}\right)^2 dx$$

Note: To use the shell method, as we are rotating around a horizontal axis, we would need to write an integral over $0 \leq y \leq \frac{1}{2}e^{-1}$, where $\frac{1}{2}e^{-1}$ is the global maximum of $f(x)$ on the interval $[0, \infty)$. This requires us to solve the equation $y = f(x)$ for x in terms of y. But we cannot solve the equation $y = xe^{-2x}$ for x, so we cannot write an integral using the shell method here.

Answer: $\int_0^\infty \pi \left(xe^{-2x}\right)^2 dx$

b. [6 points] The **area** of the region $\mathcal{R}$ (not the volume of the rotated solid) is given by the improper integral

$$\int_0^\infty xe^{-2x}\,dx.$$

Determine whether this improper integral is convergent or divergent.
You may use either a direct computation or the comparison test to reach your conclusion.
Fully justify your answer including using **proper notation.** Circle your final answer choice.

Circle one: **Convergent** **Divergent**

Solution: There are two ways to solve this problem:

Solution 1 (Direct computation)**:** We can directly evaluate this integral. To do so, we use proper notation and integrate by parts:

$$\begin{aligned}\int_0^\infty xe^{-2x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-2x}\,dx \\ &= \lim_{b\to\infty}\left(-\frac{1}{2}xe^{-2x}\Big|_0^b + \frac{1}{2}\int_0^b e^{-2x}\,dx\right) \\ &= \lim_{b\to\infty}\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^b\right) \\ &= \lim_{b\to\infty}\left(-\frac{1}{2}be^{-2b} - \frac{1}{4}e^{-2b} + 0 + \frac{1}{4}\right) = \frac{1}{4}\end{aligned}$$

(The limit $\lim_{b\to\infty} be^{-2b}$ can be computed by L'Hôpital's rule or a dominating functions argument.) Therefore, by **direct computation,** the integral is **convergent.**

Solution 2 (Comparison test)**:** Alternatively, we can show that the integral is convergent without computing its value. Observe that $x \le e^x$ for all $x \ge 0$, so that

$$xe^{-2x} \le e^x e^{-2x} = e^{-x} \qquad \text{for all } x \ge 0.$$

By the **exponential decay test,** the integral $\int_0^\infty e^{-x}\,dx$ is **convergent.**

Therefore, by the **comparison test,** the integral $\int_0^\infty xe^{-2x}\,dx$ is **convergent.**