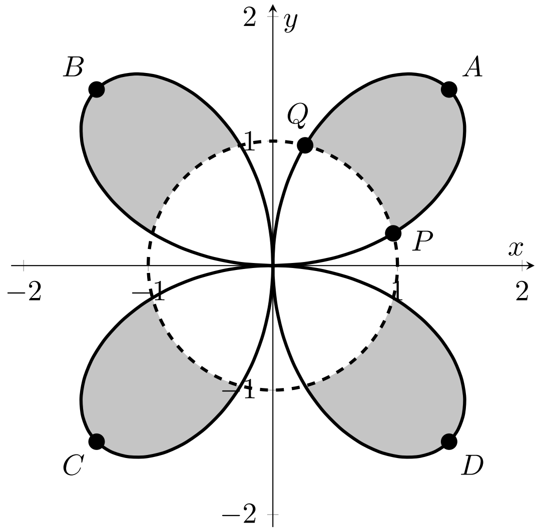
5. [12 points] A large four-leaf clover, pictured below, resides in a forest.

- The leaves of the clover are modeled by the polar curve
$$r = 2\sin(2\theta)$$
for $0 \le \theta \le 2\pi$. This is the **solid** curve in the diagram to the right.
- The polar curve
$$r = 1$$
for $0 \le \theta \le 2\pi$ is the **dashed** curve in the diagram to the right.
The leaves of the clover are light green inside of this curve, and dark green outside of it.
- All distances are measured in inches.



a. [2 points] Which of the following points labelled in the diagram above is in the portion of the polar curve $r = 2\sin(2\theta)$ traced out for $\frac{\pi}{2} \le \theta \le \pi$? Circle the **one** correct answer.
No justification is required.

Circle one: A B C $\boxed{D}$ NONE OF THESE

Solution: We substitute $\theta = \frac{3\pi}{4}$ (which satisfies $\frac{\pi}{2} \le \theta \le \pi$) into $r = 2\sin(2\theta)$ to get $r = 2\sin(\frac{3\pi}{2}) = -2$. So, the relevant polar coordinate is $(r, \theta) = (-2, \frac{3\pi}{4})$. By converting this into Cartesian coordinates using the formulas $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we get the point $(x, y) = (-2\cos(\frac{3\pi}{4}), -2\sin(\frac{3\pi}{4}))$, that is, $(x, y) = (\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}})$. This is the point D.

b. [5 points] The points P and Q, labelled above, are two intersection points of the solid and dashed curves. Write P and Q in **polar coordinates** $\boldsymbol{(r, \theta)}$**,** where $r \ge 0$ and $0 \le \theta \le 2\pi$.
Please show all of your work.

Solution: The polar coordinates (r, θ) of P and Q are both of the form $(1, \theta)$ since P and Q are on the circle $r = 1$, so we just need to find the θ-values. We must solve $2\sin(2\theta) = 1$ for θ, that is, $\sin(2\theta) = \frac{1}{2}$. We note that $\sin(x) = \frac{1}{2}$ whenever $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for an integer k. Thus $\sin(2\theta) = \frac{1}{2}$ gives the following:

$$2\theta = \frac{\pi}{6} + 2\pi k \quad \Rightarrow \quad \theta = \frac{\pi}{12} + \pi k \quad \Rightarrow \quad \theta = \frac{\pi}{12}, \frac{13\pi}{12},$$

$$2\theta = \frac{5\pi}{6} + 2\pi k \quad \Rightarrow \quad \theta = \frac{5\pi}{12} + \pi k \quad \Rightarrow \quad \theta = \frac{5\pi}{12}, \frac{17\pi}{12}.$$

The θ-values listed above are the ones which satisfy $0 \le \theta \le 2\pi$. But, since P and Q are in the first quadrant and $r > 0$, we see that P must correspond with $\theta = \frac{\pi}{12}$, and Q must correspond with $\theta = \frac{5\pi}{12}$.

Answer: P: $(r, \theta) =$ $(1, \frac{\pi}{12})$ Q: $(r, \theta) =$ $(1, \frac{5\pi}{12})$

c. [5 points] Write an expression involving **at most two integrals** that gives the **area**, in square inches, of the dark green part of the four-leaf clover. (This is the shaded region in the diagram above.) **Do not evaluate your integral(s).**

Solution: A useful formula for the area bounded by a polar curve $r = f(\theta)$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\,d\theta.$$

The points P and Q are relevant: using our θ-values from part (b), the area formula with $f(\theta) = 2\sin(2\theta)$ gives the integral

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\,d\theta, \qquad (*)$$

which is the area of the shaded region shown below to the left. We need to remove the extra area inside the dashed circle $r = 1$, shown below in the middle. Here are two possible ways to do this:

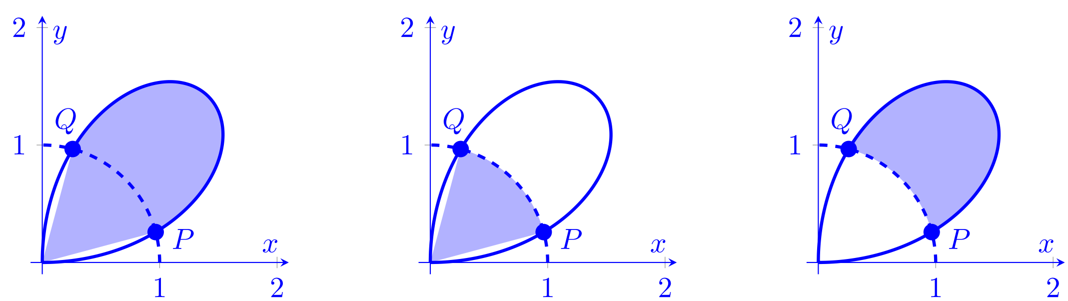
Solution 1 (Area formula)**:** The extra area inside the dashed circle can can be found using the area formula with $f(\theta) = 1$: its area is $\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1^2\,d\theta$. Subtracting this from (*), we get

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\,d\theta - \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1^2\,d\theta.$$

Solution 2 (Area of a circle)**:** Note that the area inside the dashed circle is one-sixth the area of a circle of radius 1 (since $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{2\pi}{6}$), hence has area $\frac{\pi}{6}$. Subtracting this from (*), we get

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\,d\theta - \frac{\pi}{6}.$$

Either way, we now have the area of the shaded region shown below to the right. Finally, since this is only the area on one leaf of the four-leaf clover, we multiply our expression by 4, using the symmetry of the clover, to arrive at our final answer.



Answer: $2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\,d\theta - 2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1^2\,d\theta$ **OR** $2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\,d\theta - \frac{2\pi}{3}$