7. [8 points] Consider the power series below, centered at $x = 2$:

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (x-2)^n.$$

Its radius of convergence is 4; you do not need to show this.
Find the **interval of convergence** of this power series. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

Solution: The center of convergence for this power series is 2. Since its radius of convergence is 4, we only need to check convergence at the endpoints $2 \pm 4 = -2, 6$.

At $x = 6$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (6-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3}.$$

Consider the following limit computation:

$$\lim_{n\to\infty} \frac{\frac{n+2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1} = 1 > 0.$$

Note that this limit exists and is positive.
By the ***p*-series test** ($p = 2$)**,** the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is **convergent.**
So, by the **limit comparison test,** the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$ is **convergent.**
Therefore $x = 6$ is included in our interval of convergence.

At $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-2-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-4)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-1)^n 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n.$$

Now observe that

$$\sum_{n=1}^{\infty} \left| \frac{n+2}{n^3} (-1)^n \right| = \sum_{n=1}^{\infty} \frac{n+2}{n^3}.$$

Above, we showed that the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$ is **convergent.**
So, the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n$ is **absolutely convergent** (hence convergent)**.**
Therefore $x = -2$ is included in our interval of convergence.

We conclude that the interval of convergence is $[-2, 6]$.

Solution: We present an **alternate solution** using different tests to prove convergence. As a reminder, the center of convergence for this power series is 2. Since its radius of convergence is 4, we only need to check convergence at the endpoints $2 \pm 4 = -2, 6$.

At $x = 6$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (6-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3}.$$

Note that $n + 2 \leq n + n = 2n$ for $n \geq 2$, so that

$$\frac{n+2}{n^3} \leq \frac{n+n}{n^3} \leq \frac{2n}{n^3} = \frac{2}{n^2} \qquad \text{for all } n \geq 2.$$

By the ***p*-series test** $(p = 2)$**,** the series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is **convergent.**

So, by the **direct comparison test,** the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$ is **convergent.**

Therefore $x = 6$ is included in our interval of convergence.

At $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-2-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-4)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-1)^n 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n.$$

Observe that this series is of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n, \qquad \text{where} \qquad a_n = \frac{n+2}{n^3}.$$

The sequence a_n satisfies $0 < a_{n+1} < a_n$ for all $n \geq 1$, and also $\lim_{n\to\infty} a_n = 0$.

So, by the **alternating series test,** the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n$ is **convergent.**

Therefore $x = -2$ is included in our interval of convergence.

We once again conclude that the interval of convergence is $[-2, 6]$.

Answer: $[-2, 6]$