8. [8 points]

a. [4 points] Write down the first 3 nonzero terms of the Taylor series for the function

$$S(x) = \begin{cases} \dfrac{e^{x^2} - 1}{3x^2} & x \neq 0, \\ \dfrac{1}{3} & x = 0, \end{cases}$$

centered at $x = 0$. You do not need to simplify any numbers in your answer.

Solution: We can use the "known" Taylor series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$
$$e^{x^2} - 1 = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$
$$\frac{e^{x^2} - 1}{3x^2} = \frac{1}{3} + \frac{x^2}{3 \cdot 2!} + \frac{x^4}{3 \cdot 3!} + \cdots$$

This gives the first three nonzero terms.

Answer: $\frac{1}{3} + \frac{x^2}{3 \cdot 2!} + \frac{x^4}{3 \cdot 3!}$

b. [4 points] Compute the following limit. **Fully justify** your answer including using **proper notation.**

$$\lim_{x \to 0} \frac{\int_0^x \left(e^{t^2} - 1\right) dt}{x^3}$$

Hint: Your answer from the previous part may be helpful at some point.

Solution: We start by using L'Hôpital's rule. To take the derivative of the numerator, we apply the Second Fundamental Theorem of Calculus (note that the integral cannot be evaluated directly):

$$\lim_{x \to 0} \frac{\int_0^x \left(e^{t^2} - 1\right) dt}{x^3} \overset{L'H\ \frac{0}{0}}{=} \lim_{x \to 0} \frac{e^{x^2} - 1}{3x^2}.$$

Below are two possible ways to complete the problem:

Solution 1 (Taylor series)**:** Here we use our Taylor series from part (a). From this, we have

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{3x^2} = \lim_{x \to 0} \left(\frac{1}{3} + \frac{x^2}{3 \cdot 2!} + \frac{x^4}{3 \cdot 3!} + \cdots \right) = \frac{1}{3}.$$

Solution 2 (L'Hôpital's rule)**:** Alternatively, we can apply L'Hôpital's rule again:

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{3x^2} \overset{L'H\ \frac{0}{0}}{=} \lim_{x \to 0} \frac{2xe^{x^2}}{6x} = \lim_{x \to 0} \frac{e^{x^2}}{3} = \frac{1}{3}$$

Answer: $\frac{1}{3}$