10. [9 points] Let a be a real number. Consider the following integral

$$\int_0^1 ax\ln(x)\,\mathrm{d}x$$

a. [8 points] Show that the above integral converges by using a **direct computation** and find its value in terms of a. Be sure to show your full computation, and be sure to use **proper notation**.

Solution: By definition,

$$\int_0^1 x\ln(x)\,\mathrm{d}x = \lim_{b\to 0^+}\int_b^1 x\ln(x)\,\mathrm{d}x$$

We set $u=\ln(x)$, $\mathrm{d}v = x$. Then, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, $v=\frac{x^2}{2}$. Now, integrating by parts,

$$\begin{aligned}
\lim_{b\to 0^+}\int_b^1 x\ln(x)\,\mathrm{d}x &= \lim_{b\to 0^+}\left(\frac{x^2\ln(x)}{2}\bigg|_b^1 - \int_b^1 \frac{x}{2}\,\mathrm{d}x\right)\\
&= \lim_{b\to 0^+}\left(\left(\frac{1^2\ln(1)}{2}-\frac{b^2\ln(b)}{2}\right)-\frac{x^2}{4}\bigg|_b^1\right)\\
&= \lim_{b\to 0^+}\left(-\frac{b^2\ln(b)}{2}-\left(\frac{1}{4}-\frac{b^2}{4}\right)\right)\\
&= \lim_{b\to 0^+}\frac{-\ln(b)}{\frac{2}{b^2}}-\frac{1}{4}\\
&\overset{\text{L'H}}{=} \lim_{b\to 0^+}\frac{-\frac{1}{b}}{-\frac{4}{b^3}}-\frac{1}{4}\\
&= \lim_{b\to 0^+}\frac{b^2}{4}-\frac{1}{4}\\
&= -\frac{1}{4}.
\end{aligned}$$

Therefore,

$$\int_0^1 ax\ln(x)\,\mathrm{d}x = a\int_0^1 x\ln(x)\,\mathrm{d}x = -\frac{a}{4}.$$

Answer: $\int_0^1 ax\ln(x)\,\mathrm{d}x =$ $-\frac{a}{4}$

b. [1 point] Find the value of a so that the function

$$p(x) = \begin{cases} ax\ln(x) & 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

is a probability density function (pdf).

Solution: For the function $p(x)$ to be a probability density function, it must satisfy $p(x)\geq 0$ and $\int_0^1 ax\ln(x)\,\mathrm{d}x = 1$. From part **a.**, we find that a must satisfy $-\frac{a}{4}=1$, which implies that $a=-4$.

Answer: $a =$ -4