5. [12 points] The Taylor series centered at $x = -1$ for a function $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{9^n (n!)^2}{(2n+1)!}(x+1)^{2n+1}$$

a. [7 points] Determine the **radius** of convergence of the Taylor series above. Show all of your work. You do **not** need to find the interval of convergence.

Solution: We use the ratio test to find the radius of convergence. First, we form

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{9^{n+1}((n+1)!)^2|(x+1)^{2(n+1)+1}|}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{9^n(n!)^2|(x+1)^{2n+1}|} \\ &= \frac{9^{n+1}}{9^n} \cdot \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{(2n+1)!}{(2n+3)!} \cdot \frac{|x+1|^{2n+3}}{|x+1|^{2n+1}} \\ &= 9 \cdot (n+1)^2 \cdot \frac{1}{(2n+3)(2n+2)} \cdot |x+1|^2\end{aligned}$$

Now, we evaluate the limit:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{9(n+1)^2}{(2n+3)(2n+2)}|x+1|^2 \\ &= \lim_{n\to\infty} \frac{9n^2}{4n^2}|x+1|^2 \\ &= \frac{9}{4}|x+1|^2\end{aligned}$$

The ratio test tells us that the Taylor series converges when this value is less than 1, i.e., $\frac{9}{4}|x+1|^2 < 1$. Rearranging the inequality, we find that $|x+1|^2 < \frac{4}{9}$, which implies that the radius of convergence is $\frac{2}{3}$.

Answer: $\frac{2}{3}$

b. [5 points] Find $f^{(2025)}(-1)$ and $f^{(2026)}(-1)$. You do not need to simplify your answers.

Solution: All even powers of $(x+1)$ have zero coefficient. Hence $f^{(2026)}(-1) = 0$.
On the other hand, $(x+1)^{2025}$ appears when $2n+1 = 2025$, i.e., when $n = 1012$. Thus

$$\frac{f^{(2025)}(-1)}{2025!} = \frac{9^{1012}(1012!)^2}{2025!}.$$

Rearranging and simplifying, we get

$$f^{(2025)}(-1) = 9^{1012}(1012!)^2.$$

Answer: $f^{(2025)}(-1) =$ $9^{1012}(1012!)^2$ and $f^{(2026)}(-1) =$ 0