6. [10 points] A power series centered at $x = 3$ is given by

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n(n^2+1)}(x-3)^n.$$

The radius of convergence of this power series is 5 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $3 - 5 = -2$, and $3 + 5 = 8$.

At $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n \cdot (n^2+1)}(-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}.$$

To determine the behavior of this, we use the Alternating Series Test. Set $a_n = \dfrac{2n+1}{n^2+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{n^2+1} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

and for all $n \geq 1$,

$$0 < a_{n+1} < a_n,$$

so by the Alternating Series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}$ converges.
Therefore $x = -2$ is included in the interval of convergence.

At $x = 8$, the series is

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n \cdot (n^2+1)}(5)^n = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}.$$

We can show that this series diverges using either the Direct Comparison Test or the Limit Comparison Test; here, we'll use the Direct Comparison Test.
For $n \geq 1$, $\dfrac{2n+1}{n^2+1} > \dfrac{2n}{n^2+n^2} = \dfrac{1}{n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the p-test, with $p = 1$. Therefore, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}$ diverges. This tells us that $x = 8$ is not included in the interval of convergence.
Therefore, the interval of convergence is $[-2, 8)$.

Interval of convergence: $[-2, 8)$