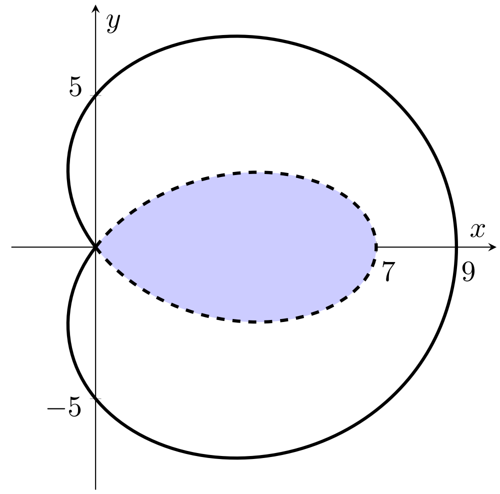
9. [12 points]

Elena, a talented landscape architect, envisions a park whose shape is defined by the polar curve $r = 5 + 8\cos(\theta) - 4\cos^2(\theta)$, as illustrated to the right. In her design, the inner loop of the curve serves as an ideal location for a lake, represented by the shaded region. The solid outer curve in the diagram represents the walking trail that winds around the park.



a. [4 points] Using the factorization $5 + 8\cos(\theta) - 4\cos^2(\theta) = (1 + 2\cos(\theta))(5 - 2\cos(\theta))$, find the values of θ in the interval $[0, 2\pi)$ for which the curve passes through the origin.

Solution: To find the values of θ in the interval $[0, 2\pi)$ where the curve passes through the origin, we set $r = 0$. Using the factorization $5+8\cos(\theta)-4\cos^2(\theta) = (1+2\cos(\theta))(5-2\cos(\theta))$, we find that $r = 0$ when

$$1 + 2\cos(\theta) = 0 \quad \text{or} \quad 5 - 2\cos(\theta) = 0$$
$$\cos(\theta) = -\frac{1}{2} \quad \text{or} \quad \cos(\theta) = \frac{5}{2}.$$

The equation $\cos(\theta) = -\frac{1}{2}$ has two solutions in the interval $[0, 2\pi)$: $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. The equation $\cos(\theta) = \frac{5}{2}$ has no solutions.

Answer: $\theta =$ $\frac{2\pi}{3}, \frac{4\pi}{3}$

b. [4 points] To determine the amount of water required to fill the lake, Elena wants to calculate the area of the surface of the lake. Write an expression involving one or more integrals that represents the area of the shaded region. Do not evaluate the integral(s).

Solution: From part **a**, we determine that the inner loop is traced for θ in the interval $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$. Using the formula for the area, the total area of the shaded region is given by

$$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2 \, d\theta.$$

Answer: $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{(5 + 8\cos(\theta) - 4\cos^2(\theta))^2}{2} \, \mathbf{d}\theta$

c. [4 points] Recall that the solid outer curve in the diagram represents the walking trail. Write an expression involving one or more integrals that represents the total length of the walking trail. Do not evaluate the integral(s).

Solution: The solid outer curve is traced for θ in the intervals $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$. By symmetry, both segments have equal lengths. Note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -8\sin(\theta) + 8\cos(\theta)\sin(\theta)$$

Using the formula for arc length, the total length of the walking trail is given by

$$2\int_0^{\frac{2\pi}{3}} \sqrt{\left(-8\sin(\theta) + 8\cos(\theta)\sin(\theta)\right)^2 + \left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2}\, \mathrm{d}\theta$$

Answer: $2\int_0^{\frac{2\pi}{3}} \sqrt{(-8\sin(\theta) + 8\cos(\theta)\sin(\theta))^2 + \left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2}\, \mathbf{d}\theta$