1. [12 points] Lara is working on homework while sitting outside when she notices a beetle crawling on her graph paper. The beetle's position t seconds after she starts watching is given by

$$\begin{cases} x(t) = (t-3)(t-5) \\ y(t) = 5\sin\left(\dfrac{\pi t}{4}\right) \end{cases}$$

where $(0,0)$ is the origin, and the units on the x and y axes are centimeters (cm).

a. [4 points] Does the beetle ever pass through the origin? Justify your answer.

Solution: For the beetle to pass through the origin, it must satisfy $x = 0$ and $y = 0$. We see that $x = 0$ only when $t = 3$ or $t = 5$. We have $y(3) = 5\sin\left(\dfrac{3\pi}{4}\right) = \dfrac{5}{\sqrt{2}}$ and $y(5) = 5\sin\left(\dfrac{5\pi}{4}\right) = -\dfrac{5}{\sqrt{2}}$, and so we do not ever have $y = 0$ when $x = 0$. Therefore the beetle does not pass through the origin.

b. [4 points] Write an expression involving one or more integrals that gives the total distance, in cm, traveled by the beetle during the first three seconds that Lara is watching. **Do not** evaluate your integral(s). Show your work.

Solution: We have $x'(t) = 2t - 8$, and $y'(t) = \dfrac{5\pi}{4}\cos\left(\dfrac{\pi t}{4}\right)$, so during the first three seconds, the beetle travels a total of

$$\int_0^3 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^3 \sqrt{(2t-8)^2 + \left(\frac{5\pi}{4}\cos\left(\frac{\pi t}{4}\right)\right)^2}\, dt \quad \text{cm.}$$

Answer: $\displaystyle\int_0^3 \sqrt{(2t-8)^2 + \left(\frac{5\pi}{4}\cos\left(\frac{\pi t}{4}\right)\right)^2}\, dt$

c. [4 points] What is the equation of the tangent line to the beetle's path at $t = 3$? Give your answer using Cartesian coordinates (expressing y in terms of x only).

Solution: Since $x(3) = 0$, and $y(3) = \dfrac{5}{\sqrt{2}}$ (as we saw in part **a.**), we know the tangent line must pass through the point $\left(0, \dfrac{5}{\sqrt{2}}\right)$.

Using our formulas for $x'(t)$ and $y'(t)$ from part **b.**, we see that $x'(3) = -2$, and $y'(3) = \dfrac{5\pi}{4}\cos\left(\dfrac{3\pi}{4}\right) = -\dfrac{25}{4\sqrt{2}}$, and therefore the slope of the tangent line is $\frac{y'(3)}{x'(3)} = \frac{25}{8\sqrt{2}}$.

Therefore the tangent line is described by $y = \dfrac{25}{8\sqrt{2}}x + \dfrac{5}{\sqrt{2}}$.

Answer: $y = \dfrac{25}{8\sqrt{2}}x + \dfrac{5}{\sqrt{2}}$

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