2. [12 points] The Taylor series centered at $x = -2$ for a function $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{((n+1)!)^2}{2^n(2n+1)!}(x+2)^{3n+1}$$

a. [7 points] Determine the **radius** of convergence of the Taylor series above. Show all of your work. You do **not** need to find the interval of convergence.

Solution: We use the ratio test to find the radius of convergence. First, we form

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(((n+1)+1)!)^2|(x+2)^{3(n+1)+1}|}{2^{n+1}(2(n+1)+1)!} \cdot \frac{2^n(2n+1)!}{((n+1)!)^2|(x+2)^{3n+1}|} \\ &= \frac{2^n}{2^{n+1}} \cdot \frac{((n+2)!)^2}{((n+1)!)^2} \cdot \frac{(2n+1)!}{(2n+3)!} \cdot \frac{|x+2|^{3n+4}}{|x+2|^{3n+1}} \\ &= \frac{1}{2} \cdot (n+2)^2 \cdot \frac{1}{(2n+3)(2n+2)} \cdot |x+2|^3\end{aligned}$$

Now, we evaluate the limit:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(n+2)^2}{(2n+3)(2n+2)} \cdot |x+2|^3 \\ &= \lim_{n\to\infty} \frac{n^2}{8n^2}|x+2|^3 \\ &= \frac{1}{8}|x+2|^3\end{aligned}$$

The ratio test tells us that the Taylor series converges when this value is less than 1, i.e., $\frac{1}{8}|x+2|^3 < 1$. Rearranging the inequality, we find that $|x+2|^3 < 8$, which implies that the radius of convergence is 2.

Answer: 2

b. [5 points] Find $f^{(60)}(-2)$ and $f^{(61)}(-2)$. You **do not** need to simplify your answers.

Solution: Observe that the power of $(x+2)$ is always $3n+1$. If we set $3n+1=60$, we get $n=59/3$. Since there is no integer n that produces a degree 60 term, the coefficient of $(x+2)^{60}$ will be 0. Hence $f^{(60)}(-2)=0$.
On the other hand, $(x+2)^{61}$ appears when $3n+1=61$, i.e. when $n=20$. Thus

$$\frac{f^{(61)}(-2)}{61!} = \frac{((20+1)!)^2}{2^{20}(2\cdot 20+1)!}$$

Rearranging and simplifying, we get

$$f^{(61)}(-2) = \frac{61!(21!)^2}{2^{20}41!}$$

Answer: $f^{(60)}(-2) =$ 0 and $f^{(61)}(-2) =$ $\frac{61!(21!)^2}{2^{20}41!}$