5. [10 points] A power series centered at $x = -1$ is given by

$$\sum_{n=1}^{\infty} \frac{n}{3^n(n^2+1)}(x+1)^n.$$

The radius of convergence of this power series is 3 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $-1-3=-4$, and $-1+3=2$.

At $x=-4$, the series is

$$\sum_{n=1}^{\infty} \frac{n}{3^n(n^2+1)}(-4+1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

To determine the behavior of this, we use the Alternating Series Test. Set $a_n = \dfrac{n}{n^2+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

and for all $n \geq 1$,

$$0 < a_{n+1} < a_n,$$

so by the Alternating Series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges.
Therefore $x=-4$ is included in the interval of convergence.

At $x=2$, the series is

$$\sum_{n=1}^{\infty} \frac{n}{3^n(n^2+1)}(2+1)^n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}.$$

We can show that this series diverges using either the Direct Comparison Test or the Limit Comparison Test; here, we'll use the Direct Comparison Test.
For $n \geq 1$, $\dfrac{n}{n^2+1} > \dfrac{n}{n^2+n^2} = \dfrac{1}{2n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, by the p-test, with $p=1$. Therefore, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges. This tells us that $x=2$ is not included in the interval of convergence.
Therefore, the interval of convergence is $[-4, 2)$.

Interval of convergence: $[-4, 2)$