7. [9 points] Consider the function $g(x) = 3x\sin\left(\frac{x^3}{2}\right)$.

a. [5 points] Give the first three nonzero terms of the Taylor series for $g(x)$ centered about $x = 0$. Show all your work.

Solution: Using the Taylor series expansion of $\sin(y)$ with $y = \frac{x^3}{2}$ about $x = 0$, we obtain

$$\sin\left(\frac{x^3}{2}\right) = \frac{x^3}{2} - \frac{x^9}{3!2^3} + \frac{x^{15}}{5!2^5} + \cdots$$

Therefore, the Taylor series of $g(x)$ about $x = 0$ is

$$g(x) = 3x\sin\left(\frac{x^3}{2}\right) = 3x\left(\frac{x^3}{2} - \frac{x^9}{3!2^3} + \frac{x^{15}}{5!2^5} + \cdots\right) = \frac{3x^4}{2} - \frac{3x^{10}}{3!2^3} + \frac{3x^{16}}{5!2^5} + \cdots$$

Answer: $\frac{3x^4}{2} - \frac{3x^{10}}{3!2^3} + \frac{3x^{16}}{5!2^5} + \cdots$

b. [4 points] The function $g(x)$ has a continuous antiderivative, $G(x)$, with a Taylor series that converges for all x. Given that $G(0) = 2$, find the first four non-zero terms of the Taylor series for $G(x)$ centered about $x = 0$. Show all your work.

Solution: The Second Fundamental Theorem of Calculus implies

$$G(x) = \int_0^x g(t)\,\mathrm{d}t + 2.$$

Using the Taylor series expansion of $g(t)$ from part **a**, and integrating term by term, we obtain the Taylor series expansion of $G(x)$ about $x = 0$:

$$\begin{aligned} G(x) &= 2 + \int_0^x \frac{3t^4}{2}\,\mathrm{d}t - \int_0^x \frac{3t^{10}}{3!2^3}\,\mathrm{d}t + \int_0^x \frac{3t^{16}}{5!2^5}\,\mathrm{d}t + \cdots \\ &= 2 + \frac{3x^5}{2\cdot 5} - \frac{3x^{11}}{3!\cdot 2^3\cdot 11} + \frac{3x^{17}}{5!\cdot 2^5\cdot 17} + \cdots \end{aligned}$$

Answer: $2 + \frac{3x^5}{10} - \frac{3x^{11}}{3!\cdot 2^3\cdot 11} + \frac{3x^{17}}{5!\cdot 2^5\cdot 17} + \cdots$