8. [7 points] Consider functions f and g that satisfy all of the following:

- $f(x)$ is defined, continuous, and positive for all $x > 1$.
- $\lim\limits_{x\to 1^+} f(x) = \infty$ (so $f(x)$ has a vertical asymptote at $x = 1$).
- $g(x)$ is defined and differentiable for all real numbers, and $g'(x)$ is continuous.
- $F(x) = \dfrac{g(x)}{x^2-1}$ is an antiderivative of $f(x)$ for all $x > 1$,
- Some values of the functions g and g' appear in the below table.

x	1	5
$g(x)$	0	-3
$g'(x)$	-8	2

Compute the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_1^5 f(x)\,\mathrm{d}x$$

Circle one: **Diverges** $\boxed{\textbf{Converges to } \underline{\quad \frac{31}{8} \quad}}$

Solution: We start by rewriting this improper integral as a limit, and then use the First Fundamental Theorem of Calculus:

$$\begin{aligned}\int_1^5 f(x)\,\mathrm{d}x &= \lim_{b\to 1^+}\int_b^5 f(x)\,\mathrm{d}x\\ &= \lim_{b\to 1^+} F(5) - F(b)\\ &= \lim_{b\to 1^+} \frac{g(5)}{24} - \frac{g(b)}{b^2-1}\\ &= -\frac{3}{24} - \lim_{b\to 1^+}\frac{g(b)}{b^2-1}.\end{aligned}$$

As $\lim\limits_{b\to 1^+} g(b) = \lim\limits_{b\to 1^+} b^2 - 1 = 0$, we try to use L'Hôpital's Rule. We obtain:

$$\begin{aligned}\int_1^5 f(x)\,\mathrm{d}x &= -\frac{1}{8} - \lim_{b\to 1^+}\frac{g'(b)}{2b}\\ &= -\frac{1}{8} + \frac{8}{2} = 4 - \frac{1}{8} = \frac{31}{8}.\end{aligned}$$

Therefore, $\displaystyle\int_1^5 f(x)\,\mathrm{d}x$ converges to $\dfrac{31}{8}$.