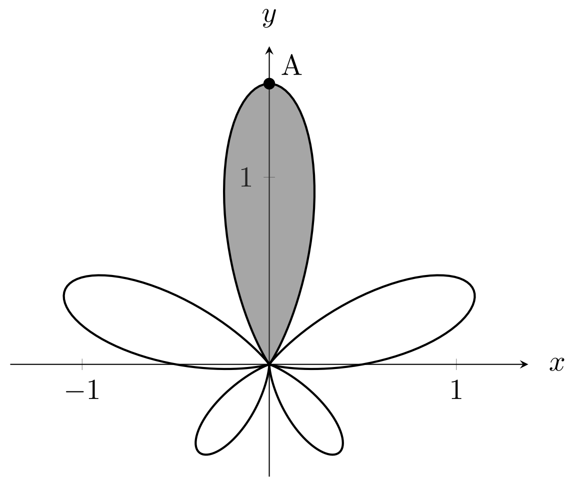
9. [10 points]

As part of a new logo design for his varsity team, Jake wants to recreate the shape of a buckeye leaf, the symbol of his home state. He models the leaf's outline, shown in the figure to the right, using the polar equation $r = \left(\frac{1}{2} + \frac{1}{2}\sin(5\theta)\right)\left(1 + \frac{1}{2}\sin(\theta)\right)$, where the leaf is traced out once as θ ranges from 0 to 2π.



a. [4 points] Find all the values of θ in the interval $[0, 2\pi)$ for which the curve passes through the origin.

Solution: To find the values of θ in the interval $[0, 2\pi)$ where the curve passes through the origin, we set $r = 0$. From the factorization $r = \left(\frac{1}{2} + \frac{1}{2}\sin(5\theta)\right)\left(1 + \frac{1}{2}\sin(\theta)\right)$, we observe that $r = 0$ when

$$\frac{1}{2} + \frac{1}{2}\sin(5\theta) = 0 \text{ or equivalently, } \sin(5\theta) = -1$$

The equation $\sin(5\theta) = -1$ has five solutions in the interval $[0, 2\pi)$: $\theta = \dfrac{3\pi}{10} + \dfrac{2n\pi}{5}$, where $n = 0, 1, 2, 3, 4$.

Answer: $\theta =$ $\dfrac{3\pi}{10}, \dfrac{7\pi}{10}, \dfrac{11\pi}{10}, \dfrac{3\pi}{2}, \dfrac{19\pi}{10}$

b. [2 points] The point A, labeled above, is the tip of the buckeye leaf. Write A in polar coordinates (r, θ), with $0 \le \theta < 2\pi$.

Solution: The point A sits on the y-axis. Hence, $\theta = \frac{\pi}{2}$. Evaluate r at $\theta = \frac{\pi}{2}$, we obtain

$$r\left(\frac{\pi}{2}\right) = \left(\frac{1}{2} + \frac{1}{2}\sin\left(5\frac{\pi}{2}\right)\right)\left(1 + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right) = \frac{3}{2}$$

Answer: $A : (r, \theta) =$ $\left(\dfrac{3}{2}, \dfrac{\pi}{2}\right)$

c. [4 points] Jake wants to paint the largest leaflet green, shown as the shaded region in the figure. Write an expression involving one or more integrals that represents the area of this leaflet. **Do not** evaluate your expression.

Solution: From part **a**, we determine that the shaded region is traced for θ in the interval $\left(\frac{3\pi}{10}, \frac{7\pi}{10}\right)$. Using the formula for the area, the total area of the shaded region is given by

$$\int_{\frac{3\pi}{10}}^{\frac{7\pi}{10}} \frac{1}{2}\left(\left(\frac{1}{2}+\frac{1}{2}\sin(5\theta)\right)\left(1+\frac{1}{2}\sin(\theta)\right)\right)^2 \, d\theta.$$

Answer: $\displaystyle\int_{\frac{3\pi}{10}}^{\frac{7\pi}{10}} \frac{1}{2}\left(\left(\frac{1}{2}+\frac{1}{2}\sin(5\theta)\right)\left(1+\frac{1}{2}\sin(\theta)\right)\right)^2 \, \mathbf{d}\theta.$