Math 216 — Second Midterm

12 November, 2012

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [16 points] Find real-valued solutions to each of the following, as indicated.

a. [8 points] Find the general solution to $y'' + 4y' + 5y = 3x - e^{-2x}$. The general solution to $y'' + 4y' + 5y = 0$ is $y = c_1e^{-2x}\cos(x) + c_2e^{-2x}\sin(x)$.

b. [8 points] Find the solution to $y'' - y = 4e^x + 3\cos(x)$, $y(0) = 0$, $y'(0) = 1$. The general solution to $y'' - y = 0$ is $y = c_1e^x + c_2e^{-x}$.

2. [14 points] Use the eigenvalue method to find a real-valued general solution to the matrix equation $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ if $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}$.

3. [12 points] Consider the system

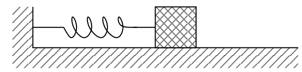
$$x_1' = a\,x_1 + 2x_2$$
$$x_2' = 4x_1 + x_2$$

(where a is a constant) and the linear constant-coefficient differential equation given in operator form by $(D^2 - 4D - 5)[y] = 0$.

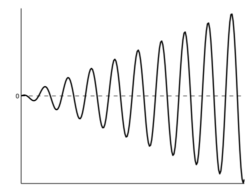
a. [6 points] If the system and differential equation are equivalent, what is the value of a?

b. [6 points] The general solution to $(D^2 - 4D - 5)[y] = 0$ is $y = c_1e^{5t} + c_2e^{-t}$. Using this, what is the general solution to the given system of differential equations?

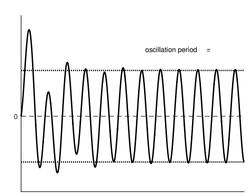
4. [12 points] Consider the differential equation $x'' + a\,x' + b\,x = A_0 \cos(\omega t)$, modeling dispacement x of the mass in the mass-spring system shown to the right. In this equation, a, b, A_0 and ω are constant parameters.



a. [6 points] If a representative graph of x as a function of time t is shown in the figure to the right, can you determine if any of a, b, A_0 or ω must be zero or must be non-zero? Must any of a, b, A_0 or ω be related in any way? Can you tell what value any of them must have?



b. [6 points] If a representative graph of x as a function of time t is shown in the figure to the right, can you determine if any of a, b, A_0 or ω must be zero or must be non-zero? Must any of a, b, A_0 or ω be related in any way? Can you tell what value any of them must have?



5. [16 points] Identify each of the following as true or false. Give a one-sentence explanation for your response in each case.

a. [4 points] Euler's method applied to the system $\mathbf{x}' = \begin{pmatrix} t & 0 \\ 1 & t^2 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives, after 2 steps with $h = 0.5$, $\mathbf{x}(1) \approx \begin{pmatrix} 0 \\ 1.25 \end{pmatrix}$.

True False

b. [4 points] Given that $\mathbf{x}_1 = \begin{pmatrix} e^t \\ 3e^t \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 2e^t \\ 6e^t \end{pmatrix}$ are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for some 2×2 matrix $\mathbf{A}$, a general solution is $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

True False

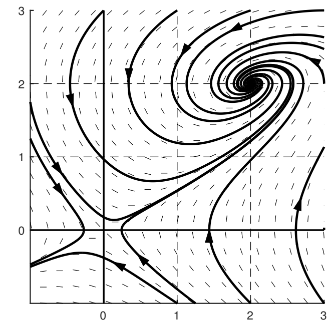
c. [4 points] If $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\ldots$, $\mathbf{x}_n(t)$ are solutions to a system of n linear first-order differential equations, and if $\mathbf{x}_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\mathbf{x}_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $\mathbf{x}_n(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, then a general solution to the system is given by $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$.

True False

d. [4 points] If one or more of the eigenvalues of the constant matrix $\mathbf{A}$ are zero, the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has no solution.

True False

6. [15 points] Consider the phase portrait shown to the right, which is for a system $x' = f(x, y)$, $y' = g(x, y)$.



a. [3 points] Explain why this system must be nonlinear.

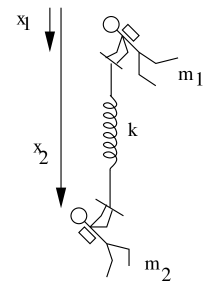
b. [8 points] Now suppose that the system is given by

$$x' = a\,y - y^2$$
$$y' = x + b\,y$$

What are the parameters a and b? Why?

c. [4 points] For each of the critical points of the system shown in the figure above, indicate whether it appears to be unstable, stable, or asymptotically stable; and whether it is a node, saddle point, center or spiral point.

7. [15 points] The figure to the right shows two (hypothetical) skydivers, with a spring connecting them. We assume that the mass of the first, m_1, is less than the mass of the second, m_2. The distances that each has fallen are x_1 and x_2, and the spring constant is k. Let L be the equilibrium length of the spring. Then the system is modeled as



$$x_1'' = \frac{k}{m_1}(-x_1 + x_2) + \left(g - \frac{kL}{m_1}\right)$$
$$x_2'' = \frac{k}{m_2}(x_1 - x_2) + \left(g + \frac{kL}{m_2}\right).$$

a. [3 points] If we write this as a matrix equation $\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{f}$, what are $\mathbf{x}$, $\mathbf{A}$ and $\mathbf{f}$?

b. [4 points] Now suppose that we're interested in finding the solution to the homogeneous problem associated with this system. If we take $\mathbf{x} = \mathbf{v}e^{\omega t}$, what equation must $\mathbf{v}$ and ω satisfy? How are $\mathbf{v}$ and ω related to the matrix $\mathbf{A}$ that you found above?

c. [8 points] Now suppose that the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ you found in (a) are $\lambda_1 = 0$, with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -4$ with $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Write the complementary homogeneous solution to your system.