This sample exam is provided to serve as one component of your studying for this exam in this course. Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [16 points] Find real-valued solutions to each of the following, as indicated.

  a. [8 points] Find the general solution to \( y'' + 4y' + 5y = 3x - e^{-2x} \). The general solution to \( y'' + 4y' + 5y = 0 \) is \( y = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x) \).

  b. [8 points] Find the solution to \( y'' - y = 4e^x + 3\cos(x) \), \( y(0) = 0 \), \( y'(0) = 1 \). The general solution to \( y'' - y = 0 \) is \( y = c_1 e^x + c_2 e^{-x} \).
2. [14 points] Use the eigenvalue method to find a real-valued general solution to the matrix equation $x' = Ax$ if $A = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}$. 
3. [12 points] Consider the system
\[
\begin{align*}
x_1' &= a \cdot x_1 + 2x_2 \\
x_2' &= 4x_1 + x_2
\end{align*}
\]
(where \(a\) is a constant) and the linear constant-coefficient differential equation given in operator form by \((D^2 - 4D - 5)[y] = 0\).

a. [6 points] If the system and differential equation are equivalent, what is the value of \(a\)?

b. [6 points] The general solution to \((D^2 - 4D - 5)[y] = 0\) is \(y = c_1 e^{5t} + c_2 e^{-t}\). Using this, what is the general solution to the given system of differential equations?
4. [12 points] Consider the differential equation $x'' + ax' + bx = A_0 \cos(\omega t)$, modeling displacement $x$ of the mass in the mass-spring system shown to the right. In this equation, $a$, $b$, $A_0$ and $\omega$ are constant parameters.

a. [6 points] If a representative graph of $x$ as a function of time $t$ is shown in the figure to the right, can you determine if any of $a$, $b$, $A_0$ or $\omega$ must be zero or must be non-zero? Must any of $a$, $b$, $A_0$ or $\omega$ be related in any way? Can you tell what value any of them must have?

b. [6 points] If a representative graph of $x$ as a function of time $t$ is shown in the figure to the right, can you determine if any of $a$, $b$, $A_0$ or $\omega$ must be zero or must be non-zero? Must any of $a$, $b$, $A_0$ or $\omega$ be related in any way? Can you tell what value any of them must have?
5. [16 points] Identify each of the following as true or false. Give a one-sentence explanation for your response in each case.

a. [4 points] Euler’s method applied to the system $x' = \begin{pmatrix} t & 0 \\ 1 & t^2 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives, after 2 steps with $h = 0.5$, $x(1) \approx \begin{pmatrix} 0 \\ 1.25 \end{pmatrix}$.

   True \hspace{1cm} False

b. [4 points] Given that $x_1 = \begin{pmatrix} e^t \\ 3e^t \end{pmatrix}$ and $x_2 = \begin{pmatrix} 2e^t \\ 6e^t \end{pmatrix}$ are solutions to $x' = Ax$ for some $2 \times 2$ matrix $A$, a general solution is $x = c_1 x_1 + c_2 x_2$.

   True \hspace{1cm} False

c. [4 points] If $x_1(t), x_2(t), \ldots, x_n(t)$ are solutions to a system of $n$ linear first-order differential equations, and if $x_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $x_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, \ldots, $x_n(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, then a general solution to the system is given by $x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$.

   True \hspace{1cm} False

d. [4 points] If one or more of the eigenvalues of the constant matrix $A$ are zero, the linear system $x' = Ax$ has no solution.

   True \hspace{1cm} False
6. [15 points] Consider the phase portrait shown to the right, which is for a system \( x' = f(x, y) \), \( y' = g(x, y) \).
   a. [3 points] Explain why this system must be nonlinear.

b. [8 points] Now suppose that the system is given by

\[
\begin{align*}
x' &= ay - y^2 \\
y' &= x + by
\end{align*}
\]

What are the parameters \( a \) and \( b \)? Why?

c. [4 points] For each of the critical points of the system shown in the figure above, indicate whether it appears to be unstable, stable, or asymptotically stable; and whether it is a node, saddle point, center or spiral point.
7. [15 points] The figure to the right shows two (hypothetical) skydivers, with a spring connecting them. We assume that the mass of the first, \( m_1 \), is less than the mass of the second, \( m_2 \). The distances that each has fallen are \( x_1 \) and \( x_2 \), and the spring constant is \( k \). Let \( L \) be the equilibrium length of the spring. Then the system is modeled as

\[
x_1'' = \frac{k}{m_1}(-x_1 + x_2) + \left( g - \frac{kL}{m_1} \right)
\]

\[
x_2'' = \frac{k}{m_2}(x_1 - x_2) + \left( g + \frac{kL}{m_2} \right).
\]

a. [3 points] If we write this as a matrix equation \( \mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{f} \), what are \( \mathbf{x} \), \( \mathbf{A} \) and \( \mathbf{f} \)?

b. [4 points] Now suppose that we’re interested in finding the solution to the homogeneous problem associated with this system. If we take \( \mathbf{x} = v e^{\omega t} \), what equation must \( v \) and \( \omega \) satisfy? How are \( v \) and \( \omega \) related to the matrix \( \mathbf{A} \) that you found above?

c. [8 points] Now suppose that the eigenvalues and eigenvectors of the matrix \( \mathbf{A} \) you found in (a) are \( \lambda_1 = 0 \), with \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \lambda_2 = -4 \) with \( \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \). Write the complementary homogeneous solution to your system.