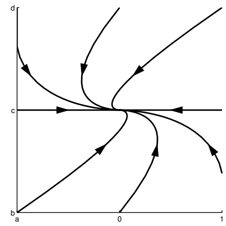
Math 216 — Final Exam

14 December, 2012

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] For each of the following the given figure is a phase portrait for a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a constant 2×2 matrix. For each select the correct characterization of the eigenvalues of $\mathbf{A}$ and fill in the requested information about an eigenvector of this matrix.

a. [4 points]

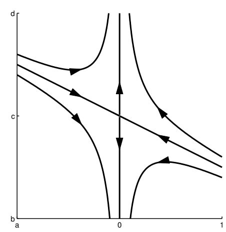


The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2$; $\lambda_1 = -1,\ \lambda_2 = 2$;
$\lambda_1 = -1,\ \lambda_2 = -2$; $\lambda_{1,2} = 1 \pm i$;
$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): ______________

b. [4 points]

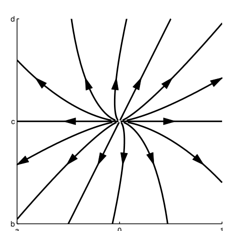


The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2$; $\lambda_1 = -1,\ \lambda_2 = 2$;
$\lambda_1 = -1,\ \lambda_2 = -2$; $\lambda_{1,2} = 1 \pm i$;
$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): ______________

c. [4 points]



The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2$; $\lambda_1 = -1,\ \lambda_2 = 2$;
$\lambda_1 = -1,\ \lambda_2 = -2$; $\lambda_{1,2} = 1 \pm i$;
$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): ______________

2. [14 points] A model for a population that is susceptible to a disease is the SI (Susceptible, Infected) model. With a few simplifying assumptions, we may model smallpox infections in a population with the SI model

$$\begin{aligned} S' &= -4SI + k(1 - S - I) \\ I' &= 4SI - I, \end{aligned}$$

where S is the fraction of the total population that is susceptible to smallpox and I is the fraction who are infected by the disease. (The remainder of the population is recovered.) We shall consider this with $k = 2$, in which case the equilibrium solutions to the system are $(S, I) = (1, 0)$ and $(S, I) = (1/4, 1/2)$.

a. [5 points] Find the linearization of this system at the critical point $(1, 0)$. Solve the linear system that you obtain.

b. [4 points] Find the linearization of this system at the critical point $(1/4, 1/2)$. Determine the type of critical point this is (that is, whether it is a node, saddle or spiral point, and its stability).

Problem 2, continued

c. [5 points] Based on your work in (a) and (b), sketch a qualitatively reasonable phase portrait (including equilibrium solutions and representative trajectories) for this system on the domain $0 \leq S \leq 1$ and $0 \leq I \leq 1$. Based on your work on this part of the problem and on (a) and (b), how do you expect the populations of susceptible and infected individuals to evolve if we start with the initial condition $(S, I) = (0.8, 0.01)$?

3. [12 points] Use Laplace transforms to solve the initial value problem

$$y'' + y = \begin{cases} 1 & t < 2 \\ 0 & t \geq 2 \end{cases}, \quad y(0) = 3, \quad y'(0) = 0.$$

4. [12 points] Fill in the blanks for each of the following inverse Laplace transforms. Show enough work to indicate how you obtained your answer.

a. [4 points] $\mathfrak{L}^{-1}\left\{\dfrac{1}{s^2+8s+16}\right\} = \mathfrak{L}^{-1}\left\{\dfrac{1}{(s+4)^2}\right\} =$ ____________________

b. [4 points] $\mathfrak{L}^{-1}\left\{\dfrac{3s+1}{(s+2)^2+16}\right\} = 3\,e^{-2t}\,\cos(4t) +$ ____________________

c. [4 points] $\mathfrak{L}^{-1}\left\{\dfrac{2-e^{-4s}}{s^2+9}\right\} = \dfrac{2}{3}\,\sin(3t) +$ ____________________

5. [14 points] Solve each of the following, as indicated.

a. [7 points] Find the general solution to $xz'(x) = z(x) + 5x^4$.

b. [7 points] Find the solution to the initial value problem $(1 + x)y' = 3y^2$, $y(0) = 2$.

6. [14 points] Consider the differential equation $\frac{d^2z}{dt} + 2\frac{dz}{dt} + z = 0$.

a. [6 points] Find a general real-valued solution to this differential equation.

b. [3 points] Rewrite the equation as a linear system of first order equations, in the form $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$.

c. [5 points] Use your solution from (a) to write the general solution for $\mathbf{x}$ in your system in (b).

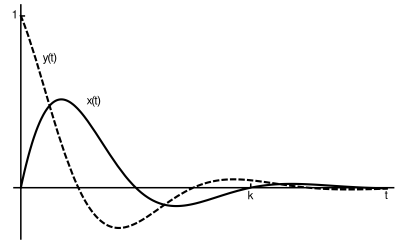
7. [10 points] Values from a cubic polynomial $f(y)$ are shown in the table below.

$y =$	-2	-1	0	1	2
$f(y) =$	-3.5	1.25	0	-1.25	3.5

a. [5 points] How many equilibrium solutions are there for the differential equation $\frac{dy}{dx} = f(y)$, where $f(y)$ is represented in the table above? What might these solutions be?

b. [5 points] Identify the stability of each of the equilibrium solutions you found in (a), and sketch a phase diagram for the differential equation.

8. [12 points] The figure to the right shows the solution to the system of first-order equations



$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & a \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where a is a constant. In the figure, x is given by the solid curve and y by the dashed curve, and k is a constant we specify later.

a. [4 points] Given these solution curves, is a positive, negative, zero, or can we not tell?

b. [4 points] If we know that $k = \pi$, what is a?

c. [4 points] Sketch a phase portrait in the x-y plane for this system, showing the solution trajectory illustrated in the figure given in the problem statement.

Do not use this page for work on the problems on the final exam. Any work you do here will not be considered when grading your exam.

Some formulas which may or may not prove useful

- Euler's, improved Euler and Runge Kutta methods iteration steps for $x' = f(t, x)$ (x and f scalar or vector):
  Euler: $x_{j+1} = x_j + hf(t_j, x_j)$
  Improved Euler: $x_{j+1} = x_j + \frac{h}{2}(f(t_j, x_j) + f(t_{j+1}, u))$, where $u = x_j + hf(t_j, x_j)$
  Runge Kutta: $x_{j+1} = x_j + \frac{h}{6}(k_1+2k_2+2k_3+k_4)$, where $k_1 = f(t_j, x_j)$, $k_2 = f(t_j+\frac{h}{2}, x_j+\frac{h}{2}k_1)$, $k_3 = f(t_j + \frac{h}{2}, x_j + \frac{h}{2}k_2)$, and $k_4 = f(t_{j+1}, x_j + hk_3)$

- Some integration formulas:
  $\int 1/\sqrt{a^2 - x^2}\, dx = \arcsin(x/a) + C$ $\quad\int 1/(a^2 + x^2)\, dx = \arctan(x/a)/a + C$
  $\int \sin^2(x)\, dx = x/2 - \sin(2x)/4 + C$ $\quad\int \cos^2(x)\, dx = x/2 + \sin(2x)/4 + C$
  $\int \tan^2(x)\, dx = \tan(x) - x + C$.

- Laplace transforms:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}

$f(t)$	$F(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$f(t) * g(t)$	$F(s) \cdot G(s)$
$-tf(t)$	$F'(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\, d\sigma$
$u(t-a)f(t-a)$	$e^{-as}F(s)$