Math 216 — First Midterm

8 October, 2012

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [14 points] Find explicit real-valued general solutions for each of the following. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $y' = 2x(e^{-x^2} - y)$

$y = \underline{\quad (x^2 + C)\, e^{-x^2} \quad}$

Solution: We use the method of integrating factors. Our equation is

$$y' + 2xy = 2xe^{-x^2},$$

so that the integrating factor is $\mu = e^{\int 2x\,dx} = e^{x^2}$. Thus we have

$$(y\, e^{x^2})' = 2x, \qquad \text{so that} \qquad y\, e^{x^2} = x^2 + C.$$

Solving for y, $y = x^2 e^{-x^2} + Ce^{-x^2}$.

b. [7 points] $y'' = -4y' - 13y$

$y = \underline{c_1 e^{-2x} \cos(3x) + c_2 e^{-2x} \sin(3x)}$

Solution: This is a linear, constant coefficient problem, so we guess $y = e^{rx}$. Then r satisfies the characteristic equation $r^2 + 4r + 13 = 0$, or $(r + 2)^2 + 9 = 0$, so $r = -2 \pm 3i$. Thus $y = c_1 e^{-2x} \cos(3x) + c_2 e^{-2x} \sin(3x)$.

2. [14 points] Solve each of the following to find explicit real-valued solutions for y. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $y' = x/(y(1+x^2))$, $y(0) = 1$.

$y =$ $\sqrt{\ln(1+x^2)+1}$

Solution: This is a nonlinear separable problem, $y\,y' = x/(1+x^2)$. Integrating both sides, we have

$$\frac{1}{2}\,y^2 = \frac{1}{2}\,\ln(1+x^2) + k, \qquad \text{so that} \qquad y = \pm\sqrt{\ln(1+x^2)+C}.$$

The initial condition $y(0) = 1$ requires that we take the positive sign for the square root and $C = 1$, so we have

$$y = \sqrt{\ln(1+x^2)+1}.$$

b. [7 points] $y'' + 14y' + 13y = 0$, $y(0) = 2$, $y'(0) = -2$.

$y =$ $2e^{-x}$

Solution: This is a linear, constant coefficient problem, so we guess $y = e^{rx}$. Then r satisfies the characteristic equation $r^2 + 14r + 13 = 0$, or $(r+1)(r+13) = 0$, so that $r = -1$ or $r = -13$. Thus the general solution is $y = c_1\,e^{-t} + c_2\,e^{-13x}$. The initial conditions require that $c_1 + c_2 = 2$ and $-c_1 - 13\,c_2 = -2$, so that $c_1 = 2$ and $c_2 = 0$. The solution is thus $y = 2e^{-x}$.

3. [8 points] A *Whiffle Ball* is a lightweight plastic ball with holes in at least one hemisphere. If we assume a viscous friction, the upward motion of a thrown or hit whiffle ball may be described in terms of its velocity v or vertical position y by $v' = -\frac{c}{m}\,v - g$ or $y'' = -\frac{c}{m}\,y' - g$. In this problem we take $c/m = 10$ and $g = 10$ (that is, approximately 9.8 m/s^2). If we start with $y(0) = 0$ and $v(0) = 5$ m/s, find the velocity v and position y of the ball.

$v =$ $6e^{-10t} - 1$

$y =$ $\frac{3}{5}\,(1 - e^{-10t}) - t$

Solution: Solving for v using the method of integrating factors, we have $v' + 10\,v = -10$, or

$$e^{10t}(v' + 10\,v) = (v\,e^{10t})' = -10e^{10t}.$$

Thus, after integrating and applying the initial condition,

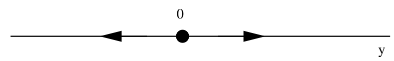
$$v\,e^{10t} = -e^{10t} + C = 6 - e^{10t}.$$

We may obtain the same solution by separating variables ($dv/(v{+}1) = 10\,dt$, so that $\ln|v{+}1| = 10t + k$, etc.). We have $v = 6e^{-10t} - 1$. Integrating to find y, we have $y = -\frac{3}{5}\,e^{-10t} - t + C$, so that, for $y(0) = 0$, $y = \frac{3}{5}(1 - e^{-10t}) - t$.

4. [15 points] Consider the differential equation $y' = y\,(y^2 + k)$.

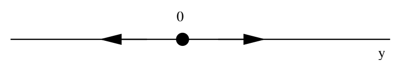
a. [4 points] If $k > 0$, find all equilibrium solutions to this equation. Determine the stability of each and draw a phase diagram.

Solution: Equilibrium solutions are found when $y' = 0$, so we need $y = 0$ or $y^2 = -k$. If $k > 0$ the latter has no (real) solutions, so the only equilibrium solution is $y = 0$. If $y < 0$, the differential equation shows $y' < 0$, and if $y > 0$, $y' > 0$. Thus this is an unstable equilibrium and the phase diagram is as shown below.



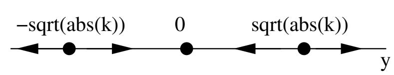
b. [4 points] If $k = 0$, find all equilibrium solutions to this equation. Determine the stability of each and draw a phase diagram.

Solution: In this case the differential equation is $y' = y^3$. Equilibrium solutions are found when $y' = 0$, so the only equilibrium solution is $y = 0$. If $y < 0$, the differential equation shows $y' < 0$, and if $y > 0$, $y' > 0$. Thus this is an unstable equilibrium and the phase diagram is the same as for part (a), as shown below.



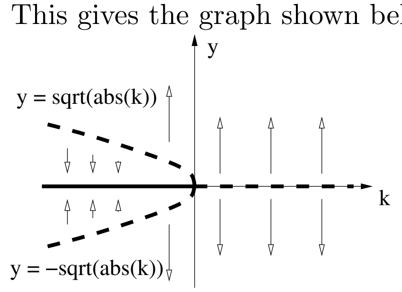
c. [4 points] If $k < 0$, find all equilibrium solutions to this equation. Determine the stability of each and draw a phase diagram.

Solution: Finally, if $k < 0$, we have $y' = y(y^2 - |k|)$. Equilibrium solutions are found when $y' = 0$, so we need $y = 0$ or $y^2 = |k|$. Thus our equilibrium solutions are $y = 0$ and $y = \pm\sqrt{|k|}$. We can write the differential equation as $y' = y(y - \sqrt{|k|})(y + \sqrt{|k|})$. Then if $y < -\sqrt{|k|}$ we see that $y' < 0$; if $-\sqrt{|k|} < y < 0$, $y' > 0$; if $0 < y < \sqrt{|k|}$, $y' < 0$, and if $y > \sqrt{|k|}$, $y' > 0$. Thus the two equilibria $y = \pm\sqrt{|k|}$ are unstable and $y = 0$ is stable, as shown in the phase diagram below.



d. [3 points] Use your work from (a)–(c) to draw a bifurcation diagram for this differential equation.

Solution: For the bifurcation diagram we graph y_{eq}, the equilibrium solutions, against the bifurcation paramter, k. This gives the graph shown below.



5. [15 points] Let $y_1 = 2\,e^{-x} + 3\,e^{2x}\cos(x)$ and $y_2 = 7\,e^{2x}\sin(x) - 4\,e^{-x}$ be solutions to a homogeneous linear constant-coefficient differential equation.

a. [9 points] Write a possible differential equation of minimal order with these solutions.

Solution: We know that roots of the characteristic equation for our differential equation are $r_1 = -1$, $r_2 = 2 + i$ and $r_3 = 2 - i$. Thus the characteristic equation is

$$(r+1)(r-(2+i))(r-(2-i)) = (r+1)(r^2-4r+5) = r^3 - 3r^2 + r + 5 = 0.$$

This corresponds to the differential equation

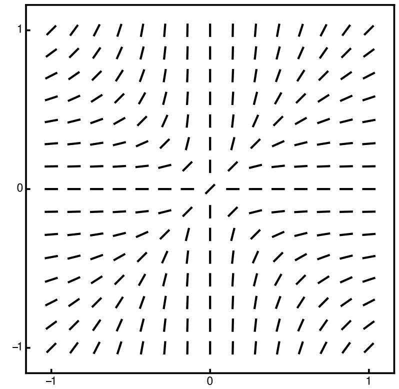
$$y''' - 3y'' + y' + 5y = 0.$$

Of course, any constant multiple of this will also work.

b. [6 points] Write the general solution to your differential equation.

Solution: We know that three linearly independent solutions to the differential equation are $y_1 = e^{-x}$, $y_2 = e^{2x}\cos(x)$ and $y_3 = e^{2x}\sin(x)$, so the general solution is

$$y = c_1 e^{-x} + c_2\,e^{2x}\cos(x) + c_3\,e^{2x}\sin(x).$$



6. [8 points] The slope field to the right is that for the differential equation $x^2y' = y^2$, which has solutions $y = x/(Cx+1)$. If we apply the initial condition $y(0) = b$, how does the number of solutions to the initial value problem depend on the value of b? Explain.

Solution: Looking only at the slope field, we note that the slopes are vertical everywhere along the y-axis other than at the origin, so we expect no solutions if $b \neq 0$. This is supported by the form of the solution, for which $y(0) = 0$ no matter what C is. If $b = 0$, we see from the slope field and differential equation that $y = x$ is a solution, and in fact that the given solution works for any value of C. Thus we expect there to be an infinite number of solutions in this case.

7. [10 points] A very simple model for the deer population P in Michigan is $P' = kP - h$, where k and h are constants, and h is the allowed number of deer that may be killed by hunters each year.

a. [4 points] What is the meaning of the parameter k? Is it positive or negative? Explain.

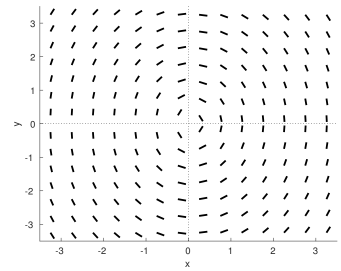
Solution: The parameter k is the difference between the birth and death rates of the deer. We expect that $k > 0$; if $k < 0$, the death rate exceeds the birth rate and the deer population will go to zero.

b. [6 points] Assume that $k > 0$ and solve the differential equation. What does your solution tell you about the long-term deer population?

Solution: We can solve by using an integrating factor or by separating variables. Separating variables, we have $(P - h/k)^{-1}P' = k$, so that $\ln|P - h/k| = kt + \hat{C}$, or $P = C\,e^{kt} + h/k$. Thus if the initial population is larger than h/k we have $C > 0$ and the population will grow to infinity; if the initial population is less than h/k, $C < 0$ and the population will go to zero in finite time.

8. [16 points] Respond to each of the following, giving a *short—one sentence* explanation of your answer. **Note:** *little partial credit will be given on this problem.*

a. [4 points] True or false: the slope field to the right corresponds to the differential equation $y' = x^2 + y^2$. Explain in one sentence.



Answer: **False**

Solution: At $(0, 1)$ the slope $y' = x^2 + y^2 = 1$, which is clearly not true for this slope field.

b. [4 points] True or false: the function $y = C\,e^{-x}$, where C is an unspecified constant, is the general solution to $y'' + 2\,y' + y = 0$. Explain in one sentence.

Answer: **False**

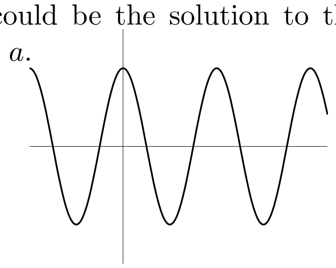
Solution: With $y = e^{rx}$ we get $r^2 + 2r + 1 = (r + 1)^2 = 0$, so the general solution is $y = C_1 e^{-x} + C_2 x e^{-x}$.

c. [4 points] True or false: if we apply Euler's method and the improved Euler method to $y' = x\,y$, $y(0) = 0$ with step-size $h = 0.1$, both predict after one step that $y(0.1) = 0$. Explain in one sentence.

Answer: **True**

Solution: Because at $(0, 1)$ we have the slope $y' = 0$, Euler's method predicts $y(0.1) = 0$; thus both slopes used in in the improved Euler method are zero, and both methods predict $y(0.1) = 0$.

d. [4 points] True or false: the graph to the right, below, could be the solution to the differential equation $y' = a^2\,y$ for some value of the constant a.



Answer: **False**

Solution: All solutions to $y' = a\,y$ are exponential, not sinusoidal.